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AUTHOR Lomon, Earle
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Abstract

These materials were developed as a practical response to some of the recommendations of the 1963 Cambridge Conference on School Mathematics (CCSM). Experimental sessions are described in detail in this report. In the Estabrook Elementary School, Lexington, Massachusetts, first grade children (1964-65 Academic Year) concentrated on material related to the real number concept. Included are descriptions of teacher and student activities. The teacher used several wooden dowels of varying length in order to involve students in discussions of the symmetric and transitive properties of inequality. In addition, the more able second grade students were also exposed to concepts and definitions for inequality, addition and subtraction, and applications to problems. The inequalities unit was also used with a pre-first grade class at Morse Elementary School in Cambridge, Massachusetts. A description of this project is provided. [Not available in hardcopy due to marginal legibility of original document.] (RP)

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CCSM
Feasibility Study No. 38

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Inequalities and Real Numbers as a Basis for School Mathematics

Earle Lomon

I. Introduction:

The report of the 1963 Cambridge Conference on School Mathematics[1] suggests that the most promising foundation for the whole mathematical curriculum is a well developed concept of the real number system. Under "The Earliest Grades, K through 2" it says, "Early experiences in studying numbers should be designed to give insight into the mathematical properties of the real number system" (p.32 of "Goals"). This recommendation, perhaps more than any other in the report, differs from the formulation of widely used mathematics curricula, both "conventional" and "modern". In the "conventional" curricula the rationals are avoided for the first few years, and negatives introduced only near the end of the elementary grades. The interelation of these systems is not shown. In "modern" curricula, the structural relations are discussed in the middle elementary grades, but emphasis there and in the earlier grades is on the cardinality of numbers as developed through set theory. A long road still awaits the student before the real number system is embedded into his mathematical language and intuition. Fractions and especially negatives still come late and a concrete feeling for their structure is not strongly built on the set theoretical beginning.

With "real number" as an opening, much of the rest of the curriculum recommended by "Goals" is roughly determined in nature and order. The rapidity of learning envisaged is largely based on having the real number system at hand, as the essential tool in most mathematical disciplines and applications. This

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[1] "Goals for School Mathematics", Houghton Mifflin Company, Boston (1963) Hereafter we will refer to this report as "Goals".

makes the recommendation one of the most debatable in the report and one of the most critical to investigate by classroom experience.

The theme has recurred in discussions since the 1963 meeting, concerning the degree of emphasis needed on the distinction between numbers as cardinals and as the related subset of the reals or "co-ordinate numbers". The degree of distinction to be given bears on the relative primary of sets, concepts and number line in the early grades. The relative strength of the intuition a child brings into school for cardinals or reals has been frequently debated. In the end these questions can only be settled with the help of much more experience in trying a carefully developed real number approach with children.

Elements desired in such an approach are listed on pp.31-33 of "Goals". (As a service to those who do not keep a copy of "Goals" in their pockets, these pages are included here as appendix.) While the 1963 CCSM meeting was in session, Professor A. Gleason tried the discussion of inequalities with young children. These were in a class of the summer session operated by ESI in the Peabody School, Cambridge. The results were encouraging. The questions raised above were given priority for the youngest group of children, when the cooperation of the staff of the Estabrook School, Lexington, was obtained to do experimental work on the proposals of "Goals".

Thus, in the Estabrook Project, first grade children (Spring 1964) and second grade children (the 1964-65 academic year) concentrated on material related to the real number concept. The classes of third to sixth grade students in the Estabrook Project also emphasized the number structure as belonging to all the reals. But the exigencies were such that the desirable base of inequalities and order structure were not developed first for those classes. At the Morse School Project (Cambridge, 1964) ideas developed at Estabrook were tried with appropriate changes, on a pre-first grade class.

The experimental sessions at the Estabrook and Morse schools will be described in detail in this report. The intent is to make our results useful to others who would like to develop this material for a larger scale experiment or for a preliminary text. We hope to make the problems of presentation, as well as of content, sufficiently evident to indicate the teacher training required. The Estabrook classes of the spring of 1964 are reported verbatim in part, relying on extensive notes and the tape recordings made of several classes. I hope that this will make the classroom context vivid to the reader, revealing the relevant attitudes and problems of young children. Perhaps this will carry over to the understanding of the other classes reported in less detail.

At all times, I have tried to work within the "discovery" method as I interpret it. Along with everyone who uses this approach, I believe that the groping of a child for an answer, or his debating it with a peer, will lead to his better understanding and remembering that answer. At the same time it develops his analytical abilities and self-confidence. However, my experience also indicates that it is important to "guide" the "discovery" to a certain extent by the questions posed, raising essential issues. This is not to say that the children's own lines of thought are not to be followed extensively, for they certainly often bring unexpected rewards. It is to be hoped that the tactics used will be clarified by the rest of this report.

Several experienced observers were present at the Morse School sessions. With the help of their notes the classroom situation is described thoroughly in Section III. The problems of teaching these younger children are given attention. The longest experiment (Estabrook, December 1964-June 1965) is reported in Section IV. There were second grade children of high ability which, together with the teacher's previous experience with the material, allowed rapid development. Less detail is reported of these classes, as the methods and material were largely as before. But these sessions allow conclusions with respect to the transferability of the material to the classroom teacher, and on the appropriate age of children for the various parts of the course.

Section V contains my general conclusions and recommendations.

II. Estabrook, Spring of 1964

With the interest and help of Mr. Cumming, Principal of Estabrook Elementary School, arrangements were made with the Lexington, Massachusetts, school authorities to permit several experimental sessions each week, with three groups of children. Teachers generously volunteered to undertake the extra planning and consultation required. In particular, Miss Marie Mortimer offered to guide the youngest group in the material of this report.

The students were an above average group of first graders, (a few were second graders) but not the most able mathematics group of their age level (mostly six years old). Two fifty minute sessions per week were planned. Miss Mortimer instructed most of the sessions, although I took over several times. We consulted on the material and its presentation one or two hours each week, in intervals of ten minutes to an hour. I was present for most of the class work. Of those sessions I missed, most were recorded on tape and one was recorded by a teacher's aide.

In the reporting of the sessions that follows, the words following a T or an S (and not in parentheses) are an approximate condensation of verbatim remarks of the teacher and student respectively. For the sake of compactness, several remarks that were in fact interlaced are often included after a single T or S. When the response of several students is being recorded, after a single S, this is denoted by a), b), etc. Descriptive remarks are in parentheses or in paragraphs not prefaced by an S or T. The presiding teacher and the date are given for each session.

Session 1 (Mortimer. Second week of February)

The teacher had about thirty wooden dowels of varying length (approximately 4"-24") and thickness (approximately 1/8"-3/8"). Strips of colored paper were fastened to each stick.

T. I have two sticks to show you today. What can you tell me about them?

S. a) One has blue paper wrapped on it, and the other has red paper. One is taller. b) One can tell it is taller because there is more paper on it. c) One looks taller. d) Put them together to see. e) Put both at the end of each other. The blue one sticks up further than the red.

T. If we want to write that down in a short way, what could we use instead of the word blue?

S. a) You could put the stick on the board. b) Make it like an equation. c) Use Bl or B. For red use R.

T. You have already used the signs +, -, = to make equations short. Close your eyes while I put down a new sign for "bigger than". Which stick is the fat end pointing at?

S. a) Looks like a sideways V b) It is pointing toward the big stick.

T. What does $B > R$ say in words?

S. Blue is taller than red.

T. One of you choose two sticks and hold them up. What can the rest of you say about them.

S. a) The purple one is smaller than the green one. b) (Writing on board) Green is taller than violet.

The teacher picks up a blue and a red stick, then puts down the blue and holds up a white stick near the red.

S. a) Blue is taller than red. b) Red is taller than white.

T. Which ones have I not compared? Which of those two is going to be taller when we check? How do you know?

S. a) The white and the blue. b) The blue would be taller than the white. Because I saw it. c) The white one was littler than the red. d) The blue one was bigger than the red so it is bigger than the white.

The teacher shows two sticks and hides a third.

S. a) Green is smaller than blue. b) (This student is allowed to see third stick, out of the sight of the class) Pink is smaller than green. c) Then pink is smaller than blue (This is checked in front of class).

Pairs of children were given sticks to compare. Then the teacher compared her stick to one of theirs, and asked how the other would compare with her stick.

Session 2. (Mortimer. Third week of February)

T. (Holding sticks of not very different length) Tell me how to compare these sticks. How do I write that?

S. a) Get them where they are the same. b) At the end. c) Use B for blue stick, R for red and the sign. (Writes B > R on board, which is then read aloud by another student. Two more sticks are compared) d) Brown is bigger than pink (writes B > P) e) Pink is smaller than brown.

T. How would you write that?

S. (Writes P < Br) Little end points to P, the little stick.

Each student took a stick and compared with a partner, each comparing from his point of view.

S. (First student uses < for "greater than" and is corrected) a) Mine is smaller than green(writes W < G). b) Mine is bigger than his (G > W).

The teacher holds up two pairs out of three sticks.

S. a) $G > Br$. b) $Br > P$.

T. Which is bigger, green or pink? Why?

S. a) (Many) Green. b) If pink is smaller than brown it must be smaller than green.

T. We are going to use a new sign (writes $\Rightarrow$) an "arrow". It looks like $=$ and $>$ put together. It tells us that from what we already know we know something else (writes $R > B,\ B > Br \Rightarrow R > Br$. She then passed out a mimeographed sheet - sheet 1 - and has them put in the missing $>$, $<$ signs, according to the size of sticks standing up in the front of the room).

Session 3. (Mortimer, March 4)

The students made many comparisons among pairs formed from a set of three sticks at the board and with pencil and paper they all developed skill in the use of $>$, $<$, and $\Rightarrow$ symbols to express their verbal statements.

Session 4. (Lomon, March 6)

The first part of the session was taken up by review questions. The class needed no help and was skillful in the comparison of sticks, the verbalization and writing of results such as

$$G < Y \Rightarrow Y > G$$

and

$$Y > G,\ G > R \Rightarrow Y > R$$

Color tagged sticks were used most of the time, but the response was equally good when a pencil, a long block and the front of a drawer (with handle) were compared.

There was confusion on the part of some students when I tried to elicit the "two stick rule" in abstract form.

T. If any stick we will call stick one is bigger than another which we will call stick two, is it sure that stick two is smaller than stick one?

S. No, there could be another stick one smaller than stick two.

I had not made it clear to some students that the use of "stick one" for any stick was nevertheless for the same stick throught the proposition. Miss Mortimer also observed that the approach to the abstraction had been too rapid, leading to responses from comparatively few students.

Session 5. (Mortimer, March 11)

The class was asked whether they thought the "two stick rule" would work for any pair of sticks. Not being able to think of a counter example, the class agreed that they would consider it to be a rule until they came across a case in which it did not work for sticks. In retrospect it seems to me that the "two stick rule" should be much less emphasized, being essentially a tautology (which the three stick rule is not). If one is bigger, by definition, the other is smaller.

Session 6. (Lomon, March 13)

T. Sometimes I will call this stick "stick one" and sometimes another. But I will keep a name for just one stick while I am talking about one problem or making one statement. When I say $A > B \Rightarrow B < A$ the A is the name of the same stick on both sides of $\Rightarrow$, and so is B. Is $A > B \Rightarrow B < A$ always true for sticks? (The children all agree, without hesitation). When we have a rule that works we can sometimes figure out something we do not know by seeing or checking. Most of you don't know my three children. My daughter Martha is taller than my son Dylan. Dylan is taller than my other daughter, Deirdre. If I don't tell you, and you have not seen them together, can you still tell me who is taller - Deirdre or Martha?

S. a) Martha's bigger than Deirdre. b) It works the same way as with sticks.

The rule may be able to save a lot of work. If I already know that the Eiffel Tower is bigger than the Empire State building (writes E.T. > E.S.)

and that the Empire State Building is bigger than the Prudential Center (E.S. $>$ P.C.) can I save myself travelling from Boston to Paris, or looking it up, to compare P.C. with E.T?

S. Yes. E.T. $>$ P.C.

About half of the class seemed convinced that the rule was useful.

Session 7. (Both teaching, March 20)

Miss Mortimer asked review questions on the two and three stick rules. The class answered well. Then I asked the following.

T. Will someone come up and compare this stick with this; and now with this?

S. a) $A < B$ b) $A < C$.

T. Now can anyone tell me if $B < C$ or $B > C$?

S. (Several chose one, several the other. They checked and found $C > B$).

T. Let us try this again (This time the third stick was shorter than the second.) We have compared two sticks with a third but we still don't have a three stick rule. What must happen before we have the rule?

The students were unresponsive to this last question and it was dropped for the time.

T. If we check and find out that $A > B$, $B > C$ and $C > D$, what can you tell me about A and D?

S. a) $A > D$ b) $D < A$.

Examples of this "four stick rule" and the similar "five stick rule" were handled well by the children. The sureness with which they could handle the concrete examples was in contrast to their difficulty with the abstract proposition.

Session 8 (Mortimer March 25)

T. Compare the blocks in the first pair and in the second pair.

S. a) $B > O$ b) $R > W$

T. (Put R on B and then listed the signs =, +, > and <)

Could any of these signs we already know be used to show that we are talking about how tall these two blocks are together?

S. a) We could write $R \nearrow B$, to show that R is put on top of B. b) We could say $B + R$.

T. If we use that last suggestion, what would we write for the height of W on top of ? Compare the height of the two piles.

S. a) $O + W$ b) $B + R > O + W$ (Other students agreed that the sentence was true).

T. The two bigger ones from each pair are taller when put together than the two smaller ones from each pair. What else can we say about the two piles? How do you know that is right?

S. a) $O + W < B + R$ b) You just have to turn everything around.

Another four blocks were used, for which the children suggested the piling and comparisons, and knew what result to expect when comparing the two piles.

T. Let us compare the heights of some of you. It would be pretty hard to stand you one on top of the other, but let us see if we know the answer anyway. Will Chris and Fred stand back to back, and also Mary and Jean?

S. a) $C > F$ b) $M > J$.

T. Now what would be true if Mary stood on Chris's head and Jean stood on Fred's head?

S. $C + M > F + J$. (The class agreed).

Session 9 (Lomon, April 3)

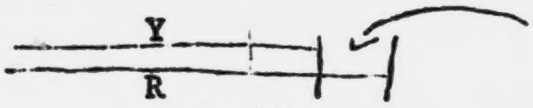
T. How do you add two sticks together? Tell me what you know about the red and yellow sticks? About the green and purple stick? About pairs of them added together?

S. (They put sticks end to end in a straight line) a) $R > Y$ b) $G > P$ c) $R + G > Y + P$

T. How would you find the difference between these two sticks?

S. a) $R > Y$ b) $Y < R$

T. You have compared the two sticks correctly. I would like to know more about them now. Show me with your hands how much bigger one is than the other (The teacher puts two sticks side by side with a pair of ends together as for comparing. A student comes up and uses his two hands to bound the distance between the other pair of ends) Very good. Can we draw the difference on the board? (After a while one student marked off the lengths of the two sticks on the board, bounding the difference with strokes



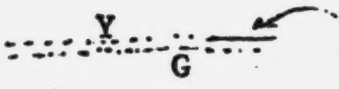
difference) .

How would we write down the difference of green and yellow?

S. $G > Y$

T. Yes, but that doesn't tell us we are talking about the size of the difference. We want something like $G + Y$ which says that we are talking about how long the two sticks are added together. Can someone mark off the difference on the board without first needing to mark off the sticks?

S. a) $G-Y$ b) (Holds the sticks together, places them on the board, and draws the chalk only along the end of the longer stick beyond the shorter stick.



difference.)

They then compared G>Y and R>B and started to find all the differences for further comparison. Keeping track of the resulting differences became difficult. The children were told that we would try to find a way of organizing all these results on one big line.

Sessions 10 and 11 (Mortimer, April 8 and 10)

The teacher suggests that they mark off their lengths on one long line she put across the blackboard. She put an x on the line and called it a starting point from which they could lay out their sticks. She pointed out that they could compare or add sticks that were not in the room at the same time. In comparing sticks, both sticks were marked off from the starting point to the right. In adding, the first stick was marked off from the starting point to the right, and the second stick was started at the end of the first. The teacher observed that the work went well.

Session 12 (Lomon, April 15)

The students made their own "stick lines" by folding paper and marking the fold with pencil. They put an x on the line, and an s below that to signify "starting point".

T. What can you do with two sticks on the stick line?

S. a) Measure them. b) Compare them.

T. Compare these on the board. How do we know which is bigger?

S. a) The one that is bigger on the stick line. b) The one that goes further from the starting point. c) The one whose mark has something left over beyond the other stick.

T. Does the end of the bigger one come closer to the door or further? (The door is to the right of the board.)

S. Closer.

T. Compare the thumb and the middle finger of your right hand at your desk.

S. I can't do it. (They all try to put their thumb and finger beside each other.)

T. Come up to the board and we will do it. (Thumb and finger are marked off on the stick line) What else do you do with sticks?

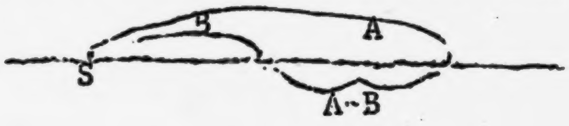
S. Add them.

The students put sticks end to end. They mark them on the stick line end after end, and also add thumb to middle finger.

T. What else can we do with sticks besides adding them?

S. Subtract them.

They find the difference by holding the sticks together. Then they work it out on the stick line as follows,



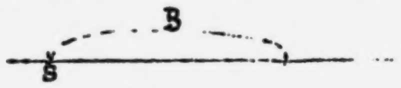
T. Can you mark it off so that A - B starts at the starting point? Think about it before we meet again.

Session 13. (Lomon, April 17)

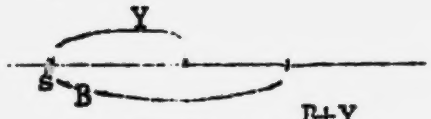
White paper and yellow and blue paper sticks (Y < B) were passed out. The children were asked to fold the white paper and draw "stick lines", with the starting points towards the left. The desk work was also demonstrated on the blackboard by students (with tagged sticks).

T. Mark the length of the blue stick. Compare the yellow with the blue. Add the yellow to the blue.

S. a)

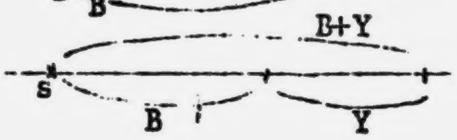


b)



B > Y

c)



T. Where was one end of B placed? In which direction did we place the other end? Where was an end of Y put? In which direction did we put the other end of Y?

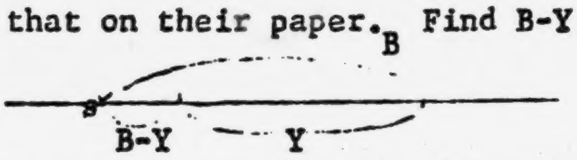
S. a) At the starting point. b) Towards the door. c) At the end of B. d) Towards the door.

T. If we want to find the difference, B-Y instead of B+Y, which way could we point the Y stick from the end of B.

S, I know! In the other direction. To the window.

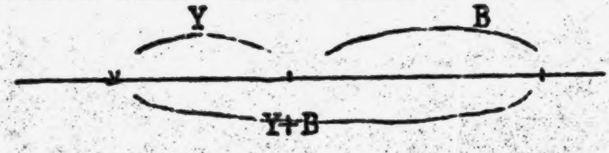
T. Everybody do that on their paper. Find B-Y.

S.



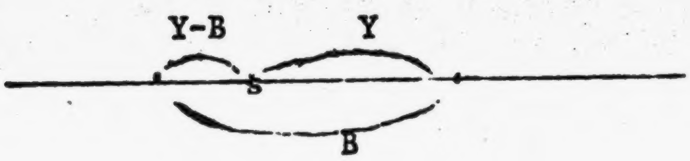
T. Draw a new "stick line" now. Mark Y on it and add B.

S.



T. Now let us subtract B. We will find Y-B instead of B-Y. Can it be done?

S. (Some students thought it could not be done, but others went right ahead).



T. What is different about Y-B from B-Y?

S. We end up on the other side of the starting point, by the same amount.

T. That is right! When we subtracted the smaller stick from the larger stick we ended up on the door side of S. When we subtracted the larger one from the smaller we ended up on the window side.

This lesson and the following ones use the "operational method" with very good results. The mathematical operations are made to correspond to mechanical operations of which every detail is prescribed. Then mathematical generalizations, such as subtracting a larger from a smaller quantity, are discovered by performing the same mechanical operations which are still found to be possible. The students surprised themselves by finding out what could be done, and they were delighted.

Session 14. (Mortimer, May 1)

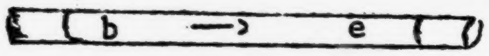
T. What am I drawing on the board? What's missing?

S. a) A stick line. b) The starting point, S.

T. I have two sticks here. When we add, do we go toward the window or the door?

S. To the door.

T. These sticks are marked with a felt pen.


colored paper band

What do you think the b on the blue stick stands for? What about the e? Where does the arrow point?

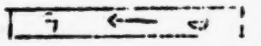
S. a) For "blue" b) I think it's for "beginning" c) The e stands for "end". d) The arrow points from the beginning to the end!

T. What about the marks on this smaller orange stick?

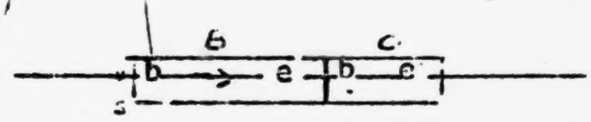
S. The arrow goes from b, which means "beginning", to e, which means "end".

T. I want us to find (writing) B+O. What do the Capitals B and O stand for? Which end of B do you think we should put at the starting point? Where does the other end go?

S. a) They stand for the blue and organge sticks. b) We would put the "beginning" at the starting point. c) The e goes towards the door.

T. Now I will add the orange stick with the "beginning" of it at the "end" of the blue (puts up ). Is the arrow right?

S. No, it should go towards the door.



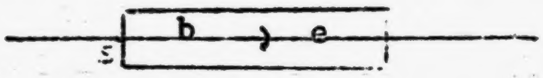
T. What should I mark for the total length? Do you all agree?

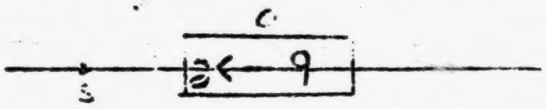
S. a) B + O b) Yes.

T. Now let us try B-O. Will someone read what that says?

S. The blue stick minus the orange.

T. Show me how to find the difference between the two sticks.

S. a) 

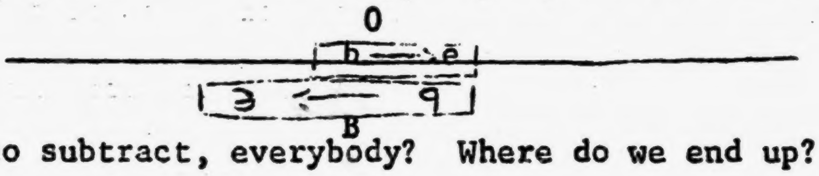
b) 

T. Does the arrow go toward the door? We have some stick left over. What do we call that part?

S. a) (many students) No! b) We call it B-O.

T. Now let us change to O-B. What do I start with? Who can subtract the blue stick? We need someone very smart.

S. a) The orange stick (marks off O readily) b) (Subtracts B according to the prescription).



T. Which way do we go to subtract, everybody? Where do we end up? Did we have enough stick?

S. a) (Chorus) To the window. b) On the other side of the starting point.

c) The orange stick wasn't big enough.

T. We had to borrow some stick. Who can put their fingers on the place where we had to borrow? Who can put down the letters that describe what was borrowed? Read it.

S. a) (Puts a finger at e of B and another at S)

b) (Writes O-B) c) That says orange minus blue.

The teacher distributed paper and "paper sticks". By folding twice the children formed three stick lines. They marked starting points near the middle.

T. Each one of you find G-Y and show me how much is left over. (The students do this successfully with some variation in positioning of sticks). Take your green stick now and mark it with b for beginning and e for end, putting an arrow in between. Do the same with the yellow. For G-Y which stick do I use first? Which way do I point it? (A student properly positioned the green stick at the board. After agreeing that this was correct the class did the same at their desks with paper sticks). Now to show -Y, do we point it towards the door or towards the window? (Again a student demonstrated at the board and then the class marked off at their desks.) What have we just found? What letters do we put down to mark it?

S. G-Y.

T. Now hold up your green and yellow sticks; holding the two ends together to compare them. Hold your fingers as far apart as the part left over and then put them down on the stick line. What is the distance the same as?

S. (The students were hesitant or confused for a few minutes, probably because the answer was too obvious.) It is the same as the distance G-Y.

Session 15. (Lomon, May 6. Tape recordings were made of this and succeeding sessions)

T. You have probably been wondering why we have been learning all these rules. We are going to play a game in which we need to know how to handle the sticks. Let's check on a few things at the board before we start the game. What kind of line have I put on the board? What do I have on the line? Where is it?

S. a) A stick line. b) A starting point. c) In the middle.

T. How long can I make my stick line? (Several hands come up). Anybody else know?

S. As long as you like.

T. Does everyone know that?

S. It can't go further than around the whole world!

T. We could get a big space ship and pull it out much further.

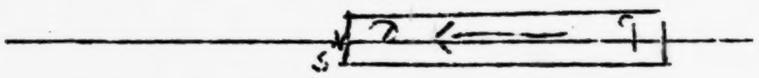
S. You would run out of chalk.

T. Are your sticks all ready?

S. a) No, they need arrows and letters. b) You mark b at the beginning and e at the end. c) The arrow goes from the beginning to the end.

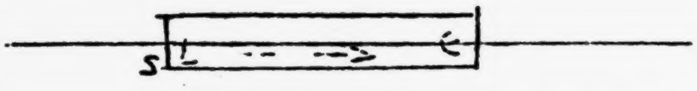
T. Who will add the blue stick to the starting point? Somebody seems to disagree. Come and do it the right way and tell us what is different.

S. a)



b) uh - oh! uh - oh!

c)



The b should be put on the s.

T. Which way does the arrow go?

S. a) Toward the window. b) No. Toward the door. c) Yes, the door.

T. One of you did it the right way and one the wrong way. Did it make any difference?

S. No. They came to the same point.

T. That is an interesting result and we will look into it on another day. Sticking to our rules for now, do we start with b or e.

S. With the beginning.

T. For adding the arrow points to ...? For subtracting...?

S. a) To the door. b) To the window. (All the class agrees).

T. You have a braintwister to think about. We started with the other end and had the arrow pointing the wrong way and he came to the same place.

S. Because he turned it around.

T. Let me turn it around but put b at s. Do I get to the same place? What was I doing by our rules?

S. a) No. b) You were subtracting.

T. Let us give names to these points. What should I call this where we added the blue stick?

S. Capital B.

T. Here I subtracted. What should I write? How many more know?

S. Blue minus blue.

T. If I take blue from blue what would I have left. Try it on the board. Where is the end? So B-B is equal to...?

S. a) Nothing. b) At the starting point. c) Zero.

T. Or nothing. Now do we have a name for taking B from the starting point. How can I write down "minus blue"?

S. First put minus, then a B for blue.

T. To make sure everyone knows which is which I will sometimes use +B for an added blue stick. Will someone please subtract the yellow from the blue?

Is he doing it correctly? Watch carefully because you will have to do the same on your sheet. What do we call it?

S. We call the part from here to here B-Y.

T. Yes. We can also call this point B-Y, because it is that far from the starting point. Someone please put up "minus yellow" for me. What do we call that point?

S. -Y.

T. Yes this path or this point can be called -Y. We are now ready for the game. We will let you work in teams. In a team everyone can help each other.

S. (They are separated into teams of four). What is this blob for? (They are looking at Sheet 3.)

T. I will tell you. You are to mark off the things listed at the top, +B +Y, +B -Y, -B +Y and so on. One or maybe more of these two stick operations will end up in the little blob. The team which finds out first wins the game. The whole team should have checked and agree, before bringing the answer to me. Any questions?

S. What do we do?

T. You take your sticks and try the things at the top of the page. One of them will bring you into the blob. Will +B +Y bring us into the blob?

S. a) No. It goes toward the door. b) Is there just one answer?

T. Not necessarily (The teacher goes around helping the teams having trouble). These two teams have come in even. You can look for another answer. Some teams have the wrong answer, so you should all check carefully. -B +Y and +Y -B are the correct answers. We will check it next time.

Session 16. (Lomon, May 8)

Class arranged in the same teams as before.

T. You will have five minutes to check your answers. Circle the ones which end up in the blob. We will check the answers later.

S. a) I finished already, b) +Y-B works.

T. Good. See if you have all the answers. The experts will show the rest of the class.

S. a) We finished ours before you did. b) They're wrong! Mine will work.

T. (Spends a few minutes helping the slow teams). Okay class. Most teams have found answers, and most have found a right one. Everybody follow while we check at the board which ones fall into the blob. What do I need on my line.

S. A starting point and a blob.

T. When we took B away from B what did we get?

S. Nothing.

T. Yes, back to the starting point we can call nothing or zero. (Marks 0 at starting point). Who has the right answer?

S. B-Y.

T. Try it at the board, everyone else at their desks.

S. (One student puts up +B and another -Y. They tell the teacher which end they are applying and in which direction they are putting the arrow).

T. What do we call the point we have ended at?

S. It is Y.

T. We have brought Y back from the end of B. If we just said -Y where would we be? (A student positions Y backwards from the starting point). Fine, that is -Y. What is the one we had before? Does it come in the blob. Try it again. Most people found it din't work, just as it didn't on the board.

S. a) +B -Y. b) I know which one works +Y-B.

T. One of you at the board and the rest at their desks. Yes, you can bring your paper to the board. How did she place +Y? Where is she to place -B?

S. a) She put b at s. b) The arrow to the door. c) (Student at board falters). She can't do it because Y is smaller then B.

T. Let's see if someone else can do it. Is she doing the right thing. (The class notes that she put the b of B at the e of Y, and the arrow to the window, and agree that this is right). It ends up in the blob as you can also see at your desks. So this is a right answer. There is another which you can check on

Session 17. (Mortimer, May 13.)

The teacher gave out new sticks (violet and red) and stick lines marked with blobs. The students tried the different combinations of two stick operations to find out which ones landed in the blob. They were in the same teams as before, and finished in about ten minutes.

T. Will someone in this team tell me one of your answers?

S. +V - R.

T. Which stick do I take first to try it? Where do I put it? Which way do I put it?

S. a) The violet stick. b) At the starting point. c) To the window.

T. How many agree it is to the window? Which stick do I take next?

S. a) (The class disagrees) No, to the door. b) The red stick.

T. Do I put it at the starting point? Which way? Where do I end up?

S. a) No, at the end of the first stick. b) To the window. c) In the blob. d) That answer is right.

T. Will someone from this team give me your other correct answer?

S. -R + V.

T. (She has the children guide her placing of sticks as before). Where do I end up?

S. (In chorus) In the blob.

T. I have written a secret in this box about the partners -R +V and +V -R. When you know the secret, raise your hand.

S. They both have - and + and V and R.

T. So does -V +R. Is it a partner too?

S. Both of the equations bring you into the blob.

T. Yes, but why do they do that? Not only do they have the same signs and letters, but they are just turned around. The first one has a +V. Who can find a (+V) in the second one? Who can find a (-R) in both? Where is the +V in the first? In the second?

S. (They find the symbols requested) a) At the beginning. b) At the end.

T. (Pointing to +R -V) Put a square box around this one and put a square box around its partner. (The students work for 3 minutes). Who can tell me what the partner is? Read both partners.

S. a) -V+R. b) +R-V, -V+R .

In order to progress to the "number line", work with "signed" sticks was stopped here. It would seem worthwhile to extend this type of work further, as a prelude to vectors. The well defined operational methods should allow a discussion of "subtracting a negative stick"; and thus be of help in the tricky probelm of (- x -), which one would like to teach by third grade. A negative stick would be one with an arrow going from e to b. Addition and subtraction would operationally be the same as for positive sticks. The b of the second stick at the e of the first; the arrow points right for addition, left for subtraction.

Session 18. (Mortimer and Lomon, May 15)

(Mortimer)

T. (Spreads a huge piece of brown paper across the blackboard. It has a long line on it, marked with ink.) We have something special to do with this giant stick line. What do we usually have on a stick line that is missing? What else have we called it?

S. a) The starting point. b) Zero.

T. Mr. Cumming asked me how much chalk I used every day. I showed him a piece this long. He said that he would like to send me a piece every Monday morning, big enough for the week, Monday to Friday. Help me figure out how big a piece I should ask him for. How do I find out how much I use in two days?

S. Put two pieces on the stick line.

T. It is hard for me to hold up two pieces together. I may only have one of this length.

S. Mark the end of the first piece, then put it down again.

T. Where do I put the first piece? Where do I put a mark? How should I mark this place?

S. a) Right here at zero. b) Put a mark at the end. c) Put a 1, because it is just one chalk and it will last one day.

T. In two days, how much do I use? Show me.

S. a) Put the mark here. b) Mark it with a 2.

The teacher goes on to 3, 4 and 5 days in the same way. She has students hold their fingers over the size of chalk needed at each stage.

T. It is Friday now. (A large fraction of the class volunteers to measure at the board.) What numeral shall I put on the end? Show me with your fingers how long a piece Mr. Cumming has to send me. What should I put down to remind us that we have added and moved toward the door?

S. a) 5. b) (The student puts fingers from 1 to 5 and is corrected by class). c) Plus.

T. Now, instead of being given some chalk, we have used up Monday's part of the five day piece. Show me on the stick line how much is left.

S. You have the piece left over and you can put it on the line.

T. Yes, but I would like to see the used up part taken off the piece for the week. We have to go to the window. You show me where to start. Good girl. With what kind of numeral do I mark this removed piece?

S. a) Cross out the 5 and put 5-1 = 4. b) Put -1 under the piece taken away.

T. Tuesday we took away what we needed. Mark that. What sign do we use?

S. a) (Correctly subtracts a unit piece from 4). b) The whole thing taken away is -2.

T. Show me where we end up Wednesday? Thursday? Friday? What signs shall I put down?

S. (Work at blackboard) a) -3. b) -4. c) -5.

T. Show me with your fingers how much would be gone. Good girl. Who can think of a name for our line? We used to call it a stick line because we measured sticks.

S. A chalk line.

T. We could call it that, but we can also measure other things. Look at what we marked on the brown line.

S. A number line.

T. Good. I had a cold last week and had to stay home for three days. Each day I used up a long handkerchief, as long as this chalk. Would someone show me how much clothesline I needed after three days. That's a girl. How much if I had been sick five days? Thank you. I have five on my line and now I take one down that I need. How much is left on the line? Good for you.

(Lomon)

T. How many know how much chalk we would need if we also came in Saturday? Where and how should I mark it?

S. a) Put another one on the end. b) Mark it at the end. c) ÷6.

T. What would happen if I wanted to go on day after day, could I keep making more and more marks or would I have to stop? Where?

S. You would have to stop. At the end of the line.

T. Show me the end. Do I have to stop there? Is there something I could do?

S. Move the zero.

T. I would lose some of my first points.

S. Put on another piece of paper.

T. Show me where I would put it. Could I get more and more paper if I need lots of numbers? Another problem now. Mr. Cumming gave us that piece of chalk that ends up at 6. But someone dropped it bring it here and broke off this dirty piece. The rest is in the cupboard. Somebody come and find out how much is in the cupboard.

S. 3¼ is left.

T. You did that in your head and you are about right. But how could you check it on the line?

S. (Measures from zero) 2½.

T. That is how much broke off. Show me how much we started with and then remove the broken part. (A student subtracts correctly. Then two other similar examples are done at the board.)

S. a) There is a day and a quarter left. b) At least one day. c) Not enough for two days.

Session 19, (Mortimer, May 20)

(Lined paper and a selection of Cuisenaire rods were distributed to the students.)

T. Which rod will we use for the unit, 1? Which one for 2? For 3? For 4? For 5?

S. a) The red ones. b) Violet. c) Green. d) Brown. e) The orange rods are for 5.

T. With a little magic I will make these rods into pieces of chalk. How many days do you think it would take us to use this piece. (She holds up the orange rod.)

S. Five days. (They correctly indentify the other four pieces.)

T. We get this (orange) piece on Monday. Do we mark it to the window or the door. Tuesday morning we want to subtract the piece we used on Monday. Where do we place it? Which way do we put it?

S. a) We add it. b) Put it toward the door. c) Put it at the five.
d) Toward the window. e) We end up at the 4.

T. It is now Wednesday morning. Subtract it again. Where do we end up. (The children respond quickly and correctly as each day's chalk is accounted. They use a finger to mark their last position.)

S. (Subtracting Friday's ration) We end at zero.

T. We are starting again with Mr. Cumming's big piece on Monday. I broke off a piece as big as the green one and it is too dirty to use. Find out how much I have left.

S. a) We end up at 2. b) at 3.

T. Remember you start from 5. (They all agree on 2.) The violet one will do for Mondays and Tuesdays. Subtract it from where we are now. Where does it bring you?

S. To zero.

T. I will borrow chalk from next door until we get our new supplies next Monday. Where do you think I ought to put this piece we use Wednesday? Where did we last end up? Which way are we going? Where do we end up?

S. a) At zero. b) To the window. c) At -1.

T. Take the red piece again and show us where we end up Thursday.

S. a) At -2. b) At -3 on Friday.

T. On Monday we get the orange piece again. Put it down so that we will know what we have left over after paying back next door. Where do we start?

S. a) At zero. b) At -3.

T. Yes, where we ended. Which way do we go? Where do you end up? How many found the same?

S. Two days (Many students express agreement.)

T. How much more should we ask Mr. Cumming to send so that we have enough for the week? Which rod?

S. (Excited volunteering) a) Three more days. b) Green.

T. Put your green and violet rods end to end. Which one would be just as long? Try it.

S. a) Brown and orange. b) The orange one.

T. We start again Monday with a weeks supply. It snows heavily Tuesday morning and there is no school all day. Take off the other days. Do we have to borrow, is there chalk left over, or do we end up at zero?

S. Chalk is left over (general agreement).

T. How much? How did you figure that out? Show me what is left.

S. a) One day. b) Because we missed a day. c) The red stick.

At this point, activity has transferred to known number line techniques with integers. It is hoped that later work with fractions will be handled more easily because the work has started from the continuous "stick line". It is also expected that subtracting from zero is seen in a less special light, because of the stick operation rules.

Session 20 (Mortimer, May 22)

In this session the Madison Project "Postman" unit was used. Checks delivered by the postman are equated with an addition to the supply of money or bank balance of the recipient. Bills brought by the postman are equated with an immediate decrease of the money supply, and all the changes are monitored on a number line. We wanted the students to have more practice in the addition and subtraction of integers. This included starting and ending at negative values, as the bank balance could be negative. In addition, we wanted to observe the transference of the ideas to representation of quantities that were not lengths, and the ability of the classroom material to be merged with that of other curricula. The students responded very well. In the next session we describe in detail their handling of "postman" material in a competitive context.

Session 21 (Mortimer, May 27)

Cuisenaire rods and paper with number lines were passed out. While Miss Mortimer directed the activity and discussion, Lomon walked around the class answering the questions of individual students.

T. Hold up the rod we use for one and tell me the color.

S. Red. (This is repeated for two, three, four and five.)

T. Where do you put the first rod on the line?

S. At the zero.

T. If I give you a check and you want to mark that, which way would you go? If I give you a bill?

S. a) To the plus side. b) To the minus side.

T. Here is a $5.00 bill. How do you mark it?

S. I start at zero and go to the minus side.

T. Pick up the three rod. Where do we start it?

S. At minus five.

T. Good! At the end of the last rod. This is a check for $3.00. Which way do we go from the end of the 5-rod?

S. a) Towards the window. b) No, towards the doors.

T. Where does the other end of the rod end up?

S. At minus two.

T. (Showing two huge stuffed, fuzzy animals.) Here are two friends of yours, Charlie and Sam. They are being sold today. Who will try to get them? I will tell you how much they cost and you have to reach there on your stick line. If I say he will be sold for $3.00, where would the rod have to end up?

S. At plus three. (Many students ready to give that answer.)

T. I am going to tell you about the checks and bills you received, and the first one who tells me, correctly, that you have reached $3.00 will get Charlie. A bill for $2.00, then a check for $3.00. Raise your hand when you think you are at +3. (One hand up.) You're not right. Now another check for $2.00. Where are you?

S. At +3.

T. She gets Charlie! Now we have to sell Sam. He will be sold for $5.00. Where does your rod have to end up?

S. At minus five.

T. Could you buy Sam with minus $5.00? Where should you end up?

S. a) No. b) At plus $5.00.

T. You get a bill for $4.00. Another bill for $1.00. Then a check for $5.00.

S. a) You are at +1. b) At zero.

T. You are back at zero because you were at -5 before. Another check for $2.00. Where are you? Now a check for $3.00.

S. a) at +2. b) Now we are at +5.

T. You win. Now I will sell Charlie for only $1.00. You first get a check for $4.00.

S. a) You have to end up at +1. b) I have enough .

T. You are right. But in this game, I have no change and you have to end up exactly right. (Goes ahead with a bill for $2.00, another bill for $2.00, a check for $1.00. Then asks last holder of Sam how much to sell him for and does so. Repeats with Charlie). Now I will change the rules. If you have more money you can buy, but you have to tell me how much change I have to give you. If I was selling Sam for $3.00 and you had $5.00, could you buy him? And what would you have to tell me.

S. a) Yes. b) That there is $2.00 in change.

T. I will sell him for $6.00. Starting at zero you get a bill for $2.00, then a check for $5.00. Another check for $3.00.

S. (Hands go up.) No change. I have just $6.00.

(Teacher sells Charlie for $4.00, a student asking for $1.00 change on $5.00. The students enjoyed the game and developed good facility with the needed number line movements.)

Session 22 (Mortimer, May 29)

The following variations of the game of the last session were played.

a) When too much was withdrawn from the bank, the banker demanded that the students sell an animal to replenish their balance. For instance, if -$3.00 was set by the banker (a debt to the bank of $3.00), the students were to raise their hands on receiving a bill that brought them to or below that mark.

b) The students were asked to observe the crossing or reaching of either of two bounds (an upper and a lower) set simultaneously. i.e. If they moved below -$3.00 they had to sell an animal to the banker. If they moved above +$2.00 they could buy an animal back. About 9/10ths of the class showed facility with this game. Some students had a discussion concerning "-5 is less than -3".

Session 23 (Mortimer, June 3)

T. We have a new stick and a new problem which we have never, never seen before. Who remembers when we were getting a long piece of chalk for each five days? One day Mr. C. brought it to me and I dropped it. The piece that broke off was longer than one but less than two, and it fell right between. I suddenly realized I needed something new on the stick line. What did I need?

S. A half a day. But we could put another half-day of chalk to it to have two days worth.

T. If we put two halves together, what is it the same as?

S. One whole.

T. Like our red rod. This is the rod I use for half a day; just for the morning. What do we have to put on our stick lines?

S. a) An "s". b) We call it zero now.

T. Who will mark up one days worth on the board (a student does so)? Now I want to find out how much I use in a half day. Who will mark that up.

S. (Subtracts a half from +1). I started from where the other one ended. I went toward the window because it is subtracting.

T. What numeral can I put there? How are we going to number it?

S. One minus two.

T. How much is that?

S. Three

T. Not quite. You would owe. How much?

S. One.

T. Now what will we call this point we reached with the 1/2 chalk?

S. One and a half.

T. This point is more than zero but less than one.

S. One half.

T. Who will mark on the board how much we need for all of Monday and half of Tuesday? Show me how much we need for one day. (Student working at board). In half a day. Where do we end up? Good. What is the name of that mark?

S. Three.

T. No. This is one and we have added another half. Show me with your hands how much we used in 1 1/2 days.

S. a) One and a half. b) (Shows interval at board).

T. Show me for two days. What is it called?

S. +2.

T. Show me for 2 1/2 days (Students points to 1 1/2). Where is your hand?

S. At 1 1/2.

T. So now show me 2 days (done). Now how much for 2 1/2 days. What do you have to do to get from 2 to 2 1/2 (done). What is the name of the point he found?

S. Two and a half.

T. What am I going to ask you to find now?

S. a) 3 1/2 . b) 3.

T. I would like 3. Is that more than 2 1/2 or less? More, so we have to add another numeral (see comments at end of session). How much more do we need to make 3.

S. 1/2.

T. Good. Put a half on. What do we have next?

S. 3 1/2.

T. Let us see how quickly we can finish the line. Put up 3 1/2. What name does that mark have?

S. a) (Marks 3 1/2). b. (Shows response from a part of the class) 3 1/2.

T. Now come and make 4 (This is done). What do you suppose I would like next? And after that?

S. a) (In chorus) 4 1/2. Take a 1/2 and put it where +4 ends. (Marked up). b) +5. c) Yes that's it. (marked on board).

T. Close your eyes for a minute. Think of a piece of long paper that goes around the room, out of the door, down the street and into the center of Lexington. If I had such a long piece of paper would I have to stop at 5? How many agree?

S. a) No. b) Because it is so long. c) You cannot go down the street because cars are in the road.

T. We will pretend there are no cars. But do we have numbers?

S. Yes.

T. We do! We have numerals (see comments below) to get us all the way into Boston. Who can think of a number bigger than 6000?

S. Infinity.

T. Yes, it never stops. What would we do with the other side of our number line?

S. Mark it to -5.

T. Would we have to stop there?

S. a) No, we could keep going as long as we want. b) Where would we get the paper? c) What happens when we wrap up the world?

T. I would like to see minus one-half marked on the stick line. What point do we get to when we take away one-half?

S. a) One minus two (apparently misreading the numeral 1/2). b) You end up at 4 1/2.

T. But if we start at zero and have to borrow, where do we end up? Are we on the plus or minus side of zero? What will I name the point?

S. a) On the minus side. b) Minus one-half.

T. What would happen if I borrowed another 1/2 a chalk from next door? (Mark it up on the board). Now I have used chalk for the morning and the afternoon. What did I use in all?

S. A whole.

T. Two halves is the same as a...?

S. a) A whole. b) One.

T. And if we have been subtracting?

S. Minus one.

T. Now it is Tuesday and I have to borrow for the morning. Who can put that on the line? (Quickly performed). What will I name it? Plus or minus.

S. a) 2 1/2, b) 1 1/2 . c) Minus.

T. What do you suppose the next point will be? Who can put it up?.

S. a) -2. b) (Marks point).

T. And after the -2? What comes between them? Put it up please.

S. a) -3 . b) -2 1/2 . c) (Marks point).

T. Show me with your hands the length from -2 1/2 to zero. From zero to 3. From zero to -1. From zero to 3 1/2. From zero to -1/2. (The students come up to the board one after the other, designating accurately). Good for you.

In the above the children expect and accept 1/2 + 1/2 = 1. It is left for session 25 to make this into an operational way of finding a half-stick. It is worth noting, consistent with an assertion often made by David Page, that if teachers use the word "numeral" they more often than not use it where "number" would be more appropriate. There are two examples above, but I have noted that result many other times with different teachers. In this case, the

the children have not been impressed by the distinction, so that no harm is done. It does indicate that one is probably better off dropping "numeral" altogether from the early vocabulary.

Session 24 (Lomon, June 5)

T. For a long time we have been watching how sticks work, finding stick rules, making stick lines, and so on. This morning we will look at something that might work differently. But first, we have to remember the stick rules. Does anyone remember the 3-stick rule?

S. You compare one stick with another, and then with a different one. Then one will end on the stick line first, starting from zero, another will come next, and the last after that.

T. Yes, there is a definite order. That is a very good way of putting it. Compare these two sticks (goes around for whispered answers). You all agreed that blue is bigger than green. How do we write that?

S. $B > G$.

T. Right. Someone told me the 2-stick rule. I don't want to concentrate on that today, but what does it say?

S. $G < B$.

T. Now compare these two sticks.

S. $B > W$.

T. (Writing $G > B$, $B > W$ on board). I separated the two statements by a comma to show we know two things. What else do we know because of the 3-stick rule? We know one more thing because of these two. What is it? (Listens to whispered answers). That's right, anybody else know something about blue and white? (Students are groaning for attention). You put the 2 and 3-stick rules together. Very good. Now the 3-stick rule says that if $G > B$ and $B > W$ we know ...?

S. $G > W$.

T. And how do we make a sign that says it follows from those two?

S. A double lined arrow.

T. Right (Writes $G > B$, $B > W \Longrightarrow G > W$). Other people told me $W < G$ which uses the 2 and 3 stick rules together. Do you think the rule is true for any 3 sticks? If the first is bigger than the second, and the second bigger than the third, will the first always be bigger than the third? (General agreement). Do you think it is true for anything we may call bigger than something else?

S. No.

T. That is very interesting. How many of you know a Japanese game called "paper, rock or scissors?" Come up and demonstrate or describe it.

S. Two people play. They put hands behind their backs. Then someone calls out "paper, rock or scissors" and then you pull out your hand showing "paper", "rock" or "scissors."

T. How do we know who wins?

S. It is something like "shoot" with even and odd.

T. Winning works differently in that game.

S. If somebody get scissors and somebody gets rock, rock wins.

T. Why is that?

S. Because rock breaks the scissors.

T. (Writes "Rock breaks scissors.") So rock wins. Those who know show the sign for rock (a clenched fist). Show the sign for scissors (spread index and middle fingers). What is the sign for paper? That's right, a stretched hand, thin like paper. If paper and scissors are put out which wins? Why?

S. a) Scissors. b) Because scissors cuts paper.

T. (Writes "scissors cuts paper"). If paper and rock are put out?

S. a) Rock wins. b) No, paper.

T. They say (writing) "paper wrap rock".

S. The rock may put a hole in the paper and fall out.

T. I have big bags of gravel at home, and the bags do pretty well. Does everybody have a partner? I will say 1, 2, 3, go. Put out your hand with one of the three signs when I say go. 1, 2, 3, go! Look at your partner and see who wins. You both put out rock, so neither of you win. Who won in your case? Why?

S. Because I had paper and he had rock. (Each pair says who won and why). Scissors can't beat scissors.

T. Scissors is scissors and red is red. To make it more like mathematics, what can I say instead of scissors beats paper?

S. Rock breaks scissors.

T. What mathematics symbols could I use for that?

S. You can put "rock is greater than paper."

T. Very good. Let us see if that will work or not. Is it fair to put (writing) $r > s$? (Assent). Bernie, if you see what is coming up, keep it to yourself for now. The words "greater than" are not important, but let us use the same symbol, the bigger end pointing to the winning object.

S. (They all ask for another play). I won. I won. I won.

T. Everyone who won put up your hands. Who lost? Who tied? I am glad that all those who tied are in pairs (Marking S and P on board). Which way does the sign go? Which wins?

S. a) Scissors. b) Turn that sign (in $r > s$) around.

T. Is scissors less than paper?

S. Turn it around again.

T. How many think it should be this way, and how many this way? (Most prefer $s > p$). How about paper and rock? How will we write that?

S. a) Paper wins. b) (The students write $p > r$ at desk, as teacher goes around class). $S > p$ means scissors beats paper. If I want to say paper beats rock, what do I write down? (This student responded slowly, but gets it). What if I wanted to say paper is beaten by scissors? (Writes $p > s$) Is this a good way of writing it?

S. a) If the paper had tough edges it might win. b) No. Paper isn't beaten by scissors.

T. Let us compare the three statements, $r > s$, $s > p$. If we could use our 3-stick rule what could we say. Over here we have $G > B$, $B > W \Rightarrow G > W$.

S. We know $p > r$.

T. We know that for the game, but is it true for the 3-stick rule?

S. It agrees with the backwards one.

T. (Missed the aptness of the last remark). What would our old 3-stick rule say?

S. $r > p$.

T. $r > s$ and $s > p$. Using the 3 stick rule what can we say about r and p. How many think it would say $r > p$? How many think $p > r$? Look back at $G > B$, $B > W$, is $W > G$?

S. No, $G > W$.

T. We could think of a green rock, blue scissors and white paper. Letting the colors stand for the things, what could we write for $r > s$ and $s > p$.

S. a) For $r > s$, put $G > B$. b) $s > p$ goes to $B > W$.

T. Now ompare what we have for the sticks with the 3-stick rule for the colors. Is everything the same so far? What did our 3-stick rule tell us?

S. $G > W$.

T. Yes. Here it means that the green rock beats the white paper. Is that true in this game? (Head shaking from class). The 3-stick rule doesn't work for the Japanese game. We can't arrange the things like 3-sticks. Why doesn't it work? If I told you that John always beat Bob, and Bob always beat Ted in a fight, does that mean John would beat Ted.

S. No.

T. Because they fight in many different ways. Many things are important. If we were comparing lengths of scissors, rock and paper, the 3-stick rule would work, but it doesn't always work when many things count. Who wants to play the game once more? (They all play).

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Although some of the class saw that "paper, scissors, or rock" contained a counter-example to the 3-stick rule, the comparison fell flat compared with my expectation. There are obvious improvements for the classroom presentation. For instance, one set of three children could hold a colored stick each, while another set of three could hold a paper, a scissors or a rock. Each set of three could then be arranged in order. One could then compare the first of each set with the third, and try arranging them in a circle. One could ask if a circle of stick holders could be arranged, by adding more sticks.

Session 25 (Lomon, June 9)

(Scissors and papers available to students. Big paper "sticks" at board).

T. Who knows what I am going to make?

S. A number line.

T. We will have some new questions and things to do with the number line. What goes on first?

S. A zero.

T. Let this piece of paper be my one-stick size. How would I mark the stick line to find 1 stick length, 2 sticks and so on. Come up to the board. No, I want to use the paper "stick" the long way. Where do I make the marks? What is this end called?

S. One.

T. Now how do I find two on here. That's right, I add on another stick. What do I mark at this point?

S. Two.

T. Now a hard question. Who can find how we get to minus one stick, or "owe" one stick? (A student subtracts the one-stick from +1). I would like you to take it from zero. (Other members of class are anxious to demonstrate). Where do I mark -1? That's right. Who can find -2 for me? Good. What do I put here?

S. Minus two.

T. Where will I be if I take away one stick from -2.

S. -3.

T. Where is my finger now? And now?

S. (Correctly.) a) -2. b) +3.

T. Now, if I put my finger here...?

S. +2.

T. Is that right? Where do you think I am?

S. a) Zero. b) -3. c) +1.

T. We are somehwere between +1 and +2. What do you say?

S. Plus one and two-fourths.

T. Yes, you know your fractions well. What do you see?

S. One and a half.

T. He is right, and one-half is two-quarters.

S. I think it is 1 3/4.

T. That is the problem you see. How are we going to tell if it is a 1/2 or 3/4? We will have to start finding a way of telling what a half-stick is, and marking the half and the quarter. (Hands up). You two save your ideas while I do something else first. We can tell whether something is bigger than or smaller than no matter what the length. How many sticks is this bigger than?

S. a) Smaller than one. b) Smaller than three. c) Smaller than two. d) It is not smaller than one, it is bigger than one.

T. We know it is smaller than two and bigger than one. (Writes < 2, > 1). Now we have to find out where 1 1/2 is and we can tell if it is bigger or smaller than that. Now tell us your idea to find a half-stick.

S. Take a stick and fold it over to the edge. It is then a half to the bend.

T. Right. Is this the way you wanted me to fold it? Where do I put this end?

S. At zero.

T. Yes, and the other end is at a half. Any other ideas?

S. Take that stick.

T. That may be half of some stick, but not of my one-stick.

S. You couldn't bend a stick.

T. Give us your idea of finding a half-stick.

S. In between those two places.

T. That may be 1/4but not 1/2.

S. Take another piece of paper the size of the one-stick, fold it in half, and then cut it.

T. What part is a half-stick long? Let's try it and see if it is the same as the other. Yes, it is.

S. Not exactly.

T. It is cut a little jagged, but it is close. (A widthwise fold - to make the stick narrow - caused some confusion, and was cut away).

T. Now we have to check to see if it is a half. How did we get two? We added one-stick to one-stick and got to 2 sticks. To get a half we do something similar. A half is a size which if we add it to itself we get one. Who can come and check? Yes, you are doing it right. It is a bit short, so we did not fold well enough. Here is a piece of paper. I want someone to cut it, so that when we add it to itself we get one. When she is finished, you will all get stick lines with 1, 2, 3 marked on but no half-points. You will cut a paper to get a half-stick and check that 1/2 + 1/2 = 1. What was she doing at the board when she moved it over?

S. a) She put down 1/2. b) She is adding it to itself. c) It comes right to one.

T. Cut your paper until it is half a stick, then check it. Don't cut your stick lines, cut your sticks (they each have two paper sticks to cut up). Do we twirl scissors or do we cut with them? You don't want to hurt your friend. When you find a half you will know you are right because when you add it to itself - once, twice like that - it will come out to one. (Both teachers go around the room, helping and checking). Two people have now added their stick to itself and checked it out. Mark them "1/2" on the stick, because we will be using them.

S. I have it!

T. The children who have it will stand, only when we have checked it. A little bit short, try again. Good for you. You have to prove it to me on the stick line. We can't know if you use both pieces, because they may not be the same. Stick to one piece. (More than a half of the students had finished by the end of the session.)

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I wanted more emphasis that the way of getting the half-stick was not critical - folding or guessing or whatever - but the checking of the definition 1/2 + 1/2 = 1 was the important aspect. The emphasis on folding obscured this to some extent. But some students did snip off- trial and error - to adjust their piece.

Session 26 (Lomon, June 16)

T. By special request we will start by playing one or two "paper, scissors or rock" games. (One or two in class don't want to play. Partners arranged. The two teachers pair off also). Get ready, get set, go! Who won?

S. Rock breaks scissors, I won. (Quick response all around).

T. Why did I win from Miss Mortimer?

S. Paper wraps rock.

T. And what is rock better than?

S. Rock is better than scissors.

T. So I write $r > s$, with the big end toward the winner. What is scissors better than?

S. Scissors is better than paper.

T. (Writes $s > p$). What is paper better than?

S. Paper is better than rock.

T. (Writes $p > r$). What is rock better than?

S. a) Better than paper. b) No. We have it already. Rock is bigger than scissors. c) We came up behind it now.

T. That is right. Are we going in circles? Who thinks we are? Not? How many don't know? Let us try it. Is $s > r$?

S. No, rock breaks scissors.

T. What does s beat?

S. $s > p$.

T. What will paper beat?

S. Rock.

T. Am I going 'round and 'round the same tree? Paper is better than rock, but I can't say that scissors is better than rock, even though scissors is better than paper. Does the 3-stick rule work?

S. No.

T. If $s > p$ and $p > r$, then with the 3-stick rule I could say $s > r$. If I had 3 sticks and found $G > B$ and $B > W$, would $W > G$?

S. No, $G > W$.

T. Here it works differently. We have passed out a stick line marked with a starting point. Remember to get 2 sticks long you have to add one to itself

1 + 1 = 2. How do we find a half-stick?

S. We fold a one-stick in half and cut.

T. How do we check that it is really a half-stick?

S. We add it to itself.

T. And what should happen?

S. It should be equal to one.

T. What is your idea (Student says something about picking a stick). But how would you know? Everything you do should have some way of being checked.

S. I am all mixed up. What do I do?

T. Cut a piece of paper so it is near a half. Then check it by adding it to itself, and see if it comes to 1. (Checks around the class). That's right. Good. It does not hurt if you don't get the right size stick the first time. If ou use both pieces, that's not adding it to itself. Use a pencil to mark the first move. (All are finished). Everybody take a pencil and mark 1/2 on their half stick. Can anybody tell me what I would do to find the point +1/2 on my stick line and mark it? I have checked that this is a half-stick long.

S. a) Find +1/2 away from the starting point. b) Put your half-stick at the starting point and point it towards the door.

T. Everybody do that. Put a mark at the end, and mark it 1/2. Who can tell me how to find the number 1 1/2?

S. a) Take a 1/2 and add it to 1.

T. What happens when we start at zero and add 1/2 to a 1/2?

S. You get 1.

T. Now to get 1 1/2?

S. a) You start at 2 (this student probably wanted to subtract 1/2 from 2). b) You add it to one and see where it ends.

T. I want you all to find 1 1/2 by adding your 1/2-stick to 1.

S. I have it.

T. Good.

The class then found the points 2 1/2, -1/2, - 1 1/2, - 2 1/2, with most of the students doing well. They were asked what one-third meant. They were not familiar with that, but knew that one-quarter was 1/2 of a 1/2, and came from dividing a unit into four equal parts.

T. One-third of a pie is the part you get when the pie is divided into three equal parts. When it is divided into two equal parts you get halves, 1/2 + 1/2=1. So for 1/3 we have ...?

S. 1/3 + 1/3 + 1/3 = 1. (Someone came to the board, folded the one-stick into thirds and tested. One more "paper, scissors or rock" game finished the session and the year.)

Comments on Section II:

The methods the children had of comparing sticks by placing them side by side, or of checking for a half stick, were operational methods that permitted a definite conclusion. It seems to me that a successful "discovery"approach depends on such self-checking operational methods. They also give the student some control of the structure of his mathematics, giving him a method for exploration.

In sessions 14 to 17, directed sticks were used. But full advantage was not taken of them, in that negative sticks (arrow from e(nd) to b(eginning)) were not used. It seems to me that this would be an appropriate and feasible way of getting at the result "minus times minus equals plus". It would imbed this idea within the larger vector properties, for possible later development.

It is important to find a counter-example, to show that the 3-stick rule is structural and not trivial. The "paper, scissors, or rock" game is the only one I have found which seems to be reasonable and simple enough to present to children. It does well, but could use some improvement in presentation, as indicated at the end of session 24.

If time had permitted, the development of fractions would have continued.

It remains to be seen if this development would obviate the general difficulties found in the classroom, with both the understanding and computation of fractional relations.

It was the opinion of the classroom teacher that the class had learned much that they would not have learned otherwise, and that it carried over into their other mathematics. She thought that at times it stimulated them, but at times some students became confused and bored. That better ways of keeping them all active and participating were needed. A great many more prepared work sheets would be a partial answer to this.

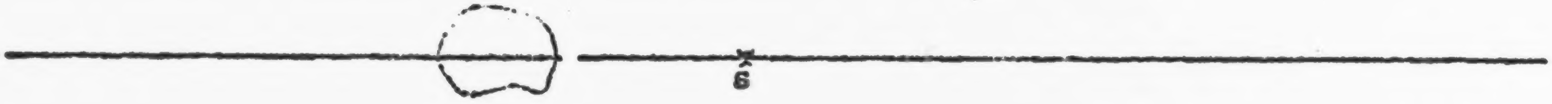
SAMPLE WORKSHEET I

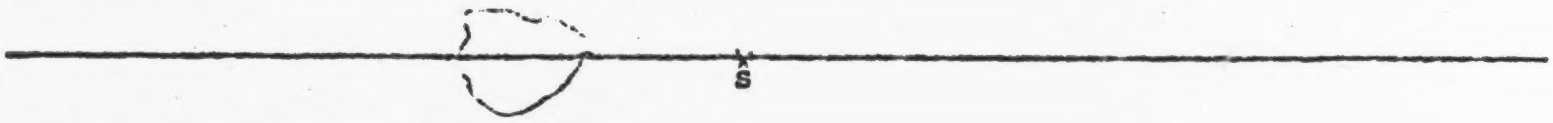
NAME ____________________

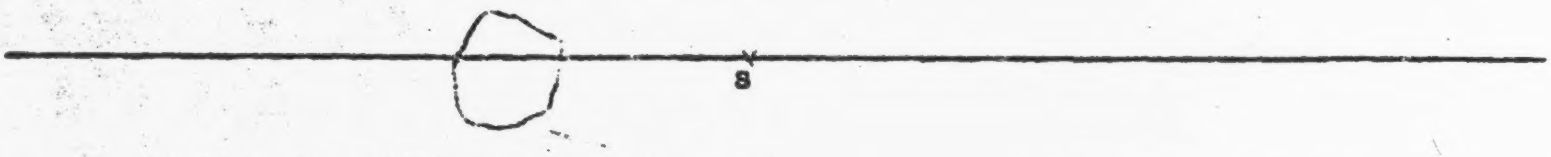
1. R G
2. G W
3. W G
4. Y P
5. B R
6. G R
7. B W
8. Y R
9. >
10. <

SAMPLE WORKSHEET II

+B + Y	+B - Y	-B + Y	-B - Y
+Y + B	+Y - B	-Y + B	-Y - B







III. Morse School Summer Session:

For five weeks in the summer of 1964, ESI staffed and directed summer classes at Morse Elementary School in Cambridge, Massachusetts. There were classes of children who would be entering, in September, every grade from first through seventh. For continuity each class had its own teacher, but most of the sessions were organized and taught by a "unit teacher" testing experimental material. This excellent research opportunity was primarily for ESS, but was used by CCSM as well. Among other mathematics units, the inequalities unit developed at Estabrook School the preceeding spring was tried with a class of fifteen pre-first grade children.

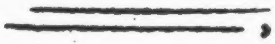
The author of this report instructed the class. The class teachers Mrs. Ruth E[illegible]s, Miss Judith Ricen, Miss Cynthia Essex, Miss Sue Powers and Dr. Marion Walter aided, took notes and discussed results and methods with the unit teacher. Fourteen sessions of 45 minutes duration were held over a four-week period. Only half the time used at Estabrook was available, but in more concentration. This gave opportunity to improve parts of the previous material and to test its validity in a different environment.

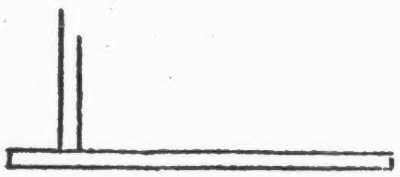
The improvement consisted of small alterations in presentation, and also in a major extension of the 3-stick (transitive) rule in its role as determinant of the order structure. The children were asked to show that the ordering of four and more sticks followed from the three-stick rule. They then applied these results to the rapid location of a given length in an array of sticks. This strongly tested the degree of deductive thinking of which those five and six-year olds were capable.

The chief environmental differences were the younger age and lesser preparation of these children compared with the Estabrook class. They were about eight months younger and had comparatively rudimentary reading and writing skills. They had fewer arithmetic skills (of little importance for the material to be tried), but had been given a small amount of number line instruction in the first week of summer school. Their attention span was shorter and their discipline less than the Estabrook children, and they were not used to having long intervals spent on one topic. Another difference was the greater frequency of sessions over a shorter time span. Even the Estabrook children seemed to forget an appreciable amount between sessions, so that the frequency was probably an advantage.

Session 1. July 7, 1964.

The dowels, with colored paper markers, were brought from Estabrook School. Two of these sticks of very different length were held up for the class to compare. The students described one as "taller" or"bigger" and the other as "smaller" or "shorter". A few children used absolutes such as "the long stick". They were then presented with a still longer stick, so that the comparative nature of the problem and terminology was stressed.

Then the students were asked to compare two sticks of only slightly different length, held far apart. The expected contradictory statements were made, and a volunteer was asked to check in any way he would like. The first student leaned them together in this way , and the next put two ends together on the desk top and examined the top ends to make the comparison.



A stick was given to each child. They were seated in pairs and a teacher was available to partner an odd child. The partner with the bigger stick was asked to stand up. In one pair both members stood up at first. Then the partner with the smaller stick was asked to raise his hand. There was no overlap or abstentions!

The teacher asked what he should write on the black board to help him remember that the red stick was longer than the green. The students suggested, in order, that he write a) "stick", b) "red" and "green", and c) R and G to denote the sticks. They then suggested using a large R and a small G. The teacher remarked that this would be difficult if there were many sticks of different size, and one had to distinguish many sizes of letters from each other.

They were familiar with the "equal" sign =, and were reminded of it by the teacher. He put the sign > on the board and asked students to point to the fat end and the thin end. He put $R > G$ on the board and asked the children to whisper in his ear if they thought they knew what it said, which several did. Then, $G < R$ was put on the board and the children guessed that this meant "green is smaller than red". The students were asked which stick did the thinner end point at, and all but two responded that it was the shorter stick. The teacher remarked that the fact that $G < R$ was true when $R > G$, and that the there was a similar result for every pair tried, could be called a "two-stick rule".

Session 2, July 8, 1964.

A pair of children, Vanja and April, were asked to compare their heights. Standing back to back, they found that Vanja was taller. The children suggested portraits to represent themselves on the board. This took a great deal of time and the result was not recognizable to the others. Then they wrote their names.

One wrote backwards, but the chief difficulty in this was that many in the class could only recognize their own name. They then used only the initial letter of their name, which was faster and better recognized by the class at least while those particular children were the subject of comparison.

$V > A$ was put on the board by the children and the others were asked what it said and why the $>$ sign pointed to April. The response was "because Vanja is taller so the fat end is towards him." They responded negatively when asked if the result could be expressed as $V < A$. Asked to put it down differently then $V > A$, but correctly (by moving the sign and the names), the response was $A < V$.

A student then compared a green and a blue stick. One child suggested that G stand for green, and another that C stand for blue. Both suggestions were accepted. One student wrote $G > C$ on the board. Another was asked to say the same thing the other way around and wrote $G < C$. Others in the class said this did not mean the same thing, and agreed on $C < G$ as the proper alternative. The students also wrote on blank paper, and put down a variety such as $\underset{G}{\overset{C}{\vee}}$.

Session 3, July 10, 1964.

Large $>$ signs had been cut out of black cardboard. A pair of children were asked to stand at the board and indicate which was taller by properly holding the sign between them. Another time the sign was placed on the blackboard (with tape) and a pair of children asked to approximately arrange themselves about it. They were asked to rearrange the sign and to themselves say, "Mary is shorter than John" in place of "John is taller than Mary".

The class was divided in two, and one half was further divided into pairs. Each pair was given a cardboard $>$ sign, and they were asked to properly arrange themselves about it. The other half of the class was asked to "read" the results and give their opinion on the truth of the statements. Then the two halves of the class exchanged roles. The use of the cardboard signs greatly facilitated the "notation" of the comparisons and allowed rapid practice of the 2-stick rule for every member of the class.

Each student was then given Worksheet I (see page 48). For the first eight problems the students were asked to fill in > or < according to the relation of the sticks that were propped up on the blackboard ledge in pairs. For the last two pairs the children were asked to put the letters approximately around the > and < signs on the worksheet. For the first problems the teacher asked a series of questions: "What do you think R stands for ?" "What kind of sign should we put in to say that red is smaller than green?" etc. The children finished the worksheet on their own. The results were examined later and many children had very good results, while some were random. Questioning of students indicated that errors seemed to be largely caused by improper identification of letters with colors and so with sticks. Their spelling was inadequate to rely on this method of notation.

Session 4, July 13, 1964.

The teacher divided the class into two teams. A stick was given to each student, and each team was instructed to keep its sticks hidden from the other team. The teacher then compared his own stick with that of a student in one of the teams. That student then wrote the relation on the board. To overcome the spelling difficulty of the previous sessions, colored chalk patches were used. The first result was notated $\overset{\text{red}}{\text{patch}}\ \circledast > T$, where T stood for "teacher's stick" and the red patch for the red stick.*

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*In this report a mark ⊛ with a color lettered above it will indicate the color of the patch, actually used without lettering.

The teacher then compared his stick with that of a student in the other team, as a result of which the student put $\overset{\text{blue}}{●} < T$ on the board, with the concurrence of his team.

The students who had seen the blue stick had not seen the red, and vice-versa. They were all asked which was bigger. The large majority thought red was bigger and this was put on the board by a student $\overset{\text{red}}{●} > \overset{\text{blue}}{●}$. The red and blue sticks were then compared directly, to check.

The procedure was repeated with another stick from each team. The result on the board was, $T > \overset{\text{green}}{●}$ $T > \overset{\text{grey}}{●}$. This time the class was divided into large groups as to which would be bigger. They were asked by the teacher if they could really tell from the facts on the board. After direct comparison $\overset{\text{green}}{●} > \overset{\text{grey}}{●}$ was listed.

A third pair of comparisons resulted in $T > \overset{\text{pink}}{●}$ $T < \overset{\text{red}}{●}$ and the class was in agreement that red was larger than pink. Some stated that they were "sure". When asked why, one replied, "I thinked it. I thinked the red is bigger than the pink, and that is how I know." They were then compared directly and $\overset{\text{red}}{●} > \overset{\text{pink}}{●}$ listed below the first two comparisons. It was expected that the children would begin to recognize the pattern of inequalities that lead to a unique result.

The use of color patches successfully overcame the notation difficulties the children had with spelling or initial letters.

Session 5, July 15, 1964.

This session was used to fortify the children's ability to recognize cases in which the transitive "3-stick rule" was applicable. They were given copies of Worksheet #3 and colored pencils to record their results in a way they could recognize a few minutes later. The box of colored sticks was kept out of sight. One student came up, chose a stick and compared it with an uncolored stick, the comparison stick. Without showing the stick to the class, he announced the result. The class was asked to use C for comparison stick and the result of their first two comparisons was

$$C > \overset{\text{yellow}}{\bullet} \qquad\qquad C > \overset{\text{red}}{\bullet}$$

The teacher asked the class if they knew what his next question would be. They replied that he would ask if the yellow or red was larger. They disagreed on which was larger and decided they would have to compare to make sure. They found, and marked, $\overset{\text{yellow}}{\bullet} > \overset{\text{red}}{\bullet}$.

Some recorded the result incorrectly, but the rest of the class told them, "The fat end is next to the bigger stick."

A second worksheet was passed out and the procedure was repeated. This time they obtained $\overset{\text{green}}{\bullet} > C \quad C > \overset{\text{yellow}}{\bullet}$. The class was nearly unanimous in the opinion that green was bigger than yellow. Asked why, one replied, "The green is bigger than the comparison stick and the comparison stick is bigger than the yellow, so the green is bigger than the yellow." They all wrote $\overset{\text{green}}{\bullet} > \overset{\text{yellow}}{\bullet}$.

The teacher remarked that they seemed to have found a rule. When they had the pattern $\overset{\text{green}}{\bullet} > C \quad C > \overset{\text{yellow}}{\bullet}$ with the comparison stick smaller than one and bigger than the other, then they knew $\overset{\text{green}}{\bullet} > \overset{\text{yellow}}{\bullet}$. This was the 3-stick rule. When the comparison stick was smaller than both, or bigger than both, there was no rule.

The session was completed by the children comparing lengths of red and black licorice; and eating the sticks after a valid comparison.

Session 6, July 17, 1964.

T. Today I will ask you to think about sticks and about comparing them. First think about a red stick smaller than a comparison stick. Next think about a green stick also smaller than the same comparison stick. Is the green or the red one bigger?

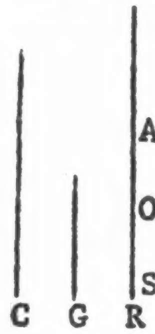
S. They could both be the same height.

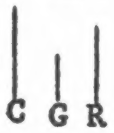
T. Could G > R ?

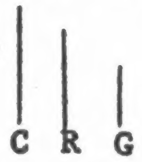
S. Could be, or R could be bigger.

T. Think of a great big comparison stick. Now think of a smaller red stick and of a green one still smaller. Start again with the very big comparison stick and now of a smaller green one and a still smaller red one. Were you able to do that? In both cases the red and green sticks were smaller than the comparison stick, but we could have either one of the colored stick bigger than the other.

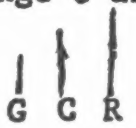
Can you think of a red stick bigger than the comparison stick? Now of a green one smaller than the comparison stick? Is the green or red smaller? Does it have to be that way? Let us see what we can draw on the board.



A student drew a line for C, one for G < C and one for R > C. Other students were challenged to do the same and yet have R < G. Several children made attempts such as 

or 

and found out which desired inequality was not fulfilled. Then one remarked, "The opposites may be fouling things up."

T. How many ways can I arrange G and R around the < sign? Which is right R < G or G < R? (He draws 

) .

That time G < R and we have found out it is always that way if we draw G < C and R > C. If it always happens it is a rule and we call it our 3-stick rule. Now if I know G > C and R > C do I have a 3-stick rule? Do we know for sure if G > R or R > G? Where does C have to be to tell us if G or R is bigger? Should $C > G$ and $C > R$, or should $C < G$ and $C < R$ or should C be in between?

S. In between .

T. If $T < G$ and $G < V$, is G between V and T? Which is bigger V or T?

S. G is in the middle.

These "thought experiments" were used to generalize the result of a few sticks. It was possible to cover cases more quickly. The need for the "comparison" stick (the one appearing in both comparisons) to be between the others in size was well brought out.

Session 7, July 20, 1964,

Two objectives were partially obtained during this session. The students observed the 4 and 5-stick rules, and thus by extension the unique ordering of many lengths. Secondly, they found that the 4-stick rule was a simple consequence of the 3-stick rule. The latter was attempted, not only so that the students would better understand the order structure, but as a general test of deductive ability in the very young.

The teacher showed the class a green, a white, and a blue stick for which $G < W < Bu$. The students were asked how the 3-stick rule would apply to these. Which would be the comparison stick? They replied that W would be compared with G and with Bu. That one would get $W > G$ and $Bu > W$, and that the 3-stick rule would say that $Bu > G$.

The teacher then showed a fourth stick, black, and asked if they could find a four stick rule. One student remarked that Bk was smaller than the other three, another that $W > Bk$. The first student added that $Bu > W$, $W > G$ and $G > Bk$. The teacher noted that in this case, with those three things true, it was also time that $Bu > G$, which was the 3-stick rule, and that $Bu > Bk$. Was the last a rule about 4 sticks? The class thought it would always be true for four sticks. The comparison steps were repeated on the board. $Bu > W, \; W > G, \; G > Bk \Longrightarrow Bu > Bk$.

The teacher said that they could check that for many sets of sticks to see if it was a rule. But on the other hand, they had already checked the 3-stick rule for many sets. Had they ever tried to show that a new rule was true just by using rules they already knew? Could anyone show that the 4-stick rule was true, by using the 3-stick rule?

The students noted that one more stick, the black one, had been added to those needed for the 3-stick rule. The teacher asked what information was needed about the black stick; and the response was $Bk < G$. The teacher put $Bu > W$, $W > G$, and $G > Bk$ on the board and asked if they could separate out the 3-stick rule. They responded that the first two relations gave $Bu > G$, using the 3-stick rule. This was noted on the board as follows:

$$\begin{array}{ccc} \underbrace{Bu > W \qquad W > G} & \qquad\qquad & G > Bk \\ \Downarrow \text{ 3-stick rule} & & \Downarrow \\ Bu > G & & G > Bk \end{array}$$

The remaining information, $G > Bk$, was repeated on the new line as shown. The teacher asked if there was another 3-stick rule left. A student said, "Yes, because black makes three." "The 3-stick rule says, that $Bu > Bk$ because $Bu > G$ and $G > Bk$." The teacher then asked them what the 4-stick rule had said. The students replied that it had said $Bu > Bk$, and that this was not new. Using the 3-stick rule twice already said $Bu > Bk$.

The teacher then passed out five paper "sticks" to each student; orange, blue, white, yellow and pink in descending lengths. He said "I have a green stick, where does it go?" By comparing it with one of their sticks, the children found in succession that it was bigger than the pink, smaller than the blue, smaller than the white, and smaller than the yellow. A couple of

2

false statements such as "It is the same as the blue," were quickly corrected by the rest of the class.

The teacher then asked if the green was between the blue and yellow? The blue and white? Is it bigger than the blue? Bigger than the orange? The students correctly answered "no" to those questions; and remarked that we already knew it, because we had checked that $G < Y$.

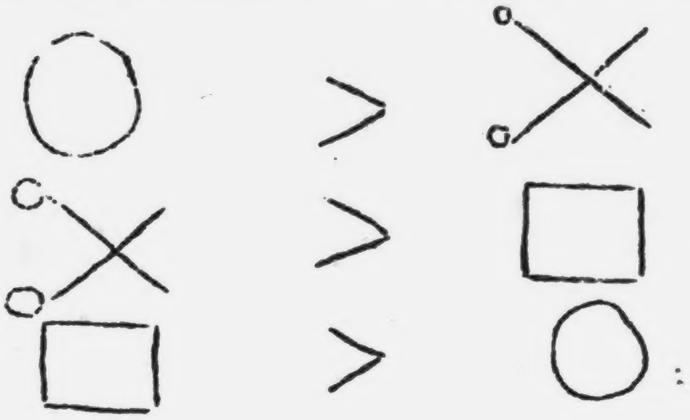
It was evident, that as one would expect, that the students knew the full structure of linear ordering, without having to break it down into an endless series of finite ordering rules. To some extent they now realized that this was determined by the 3-stick rule. The response from the class to the challenge of deducing the 4-stick rule from the 3-stick rule was satisfactory. Several members of the class were able to construct such proof and probably the majority could follow the steps.

Time did not permit the further breakdown of a finite set of lengths into various ordering rules. The efficiency of the students in the strategy game of the following sessions may have suffered from the short time spent on explicit results.

Session 8, July 21, 1964.

The first part of this lesson was devoted to establishing a counter-example to the 3-stick rule. The object was to demonstrate that such a rule was a matter of observation for specific relations of specific objects, and need not hold conceptually similar relations among other objects.

Some of the children were already familiar with the "rock, paper, scissors" game described in Section II. This game was explained to the others with the help of those who knew. The children played the game. When they had all played a round in pairs, they were asked to put up their hands, first if they had won, then if they had lost, and last if they had tied. Several individuals were asked why they had won or lost, and a complete set of results was put on the board. They agreed that one might try using > for "better than" in the same way as for "taller than", and the result was:



"rock breaks scissors"

"scissors cuts paper"

"paper wraps stone"

The students were asked if the use of > with the fat end toward the winner a good use of the sign. It was decided to see what happened if they kept using it.

For contrast, the children were asked to compare a blue, a green and a grey stick in pairs. The students marked these results on the board beside the paper, rock, scissors results.

blue ● < green ●

grey ● < blue ●

T. Which will be bigger, green or grey?

S. Green is bigger.

T. Which rule is that?

S. The three stick rule.

The answer was checked and

grey ● < green ●

added on the board. The children read "Grey is smaller than green."

T. Let us look at the first two results of the game. ○ > ✂ says rock breaks scissors, and ✂ > □ says scissors cuts paper. Does it look like we have a 3-stick rule?

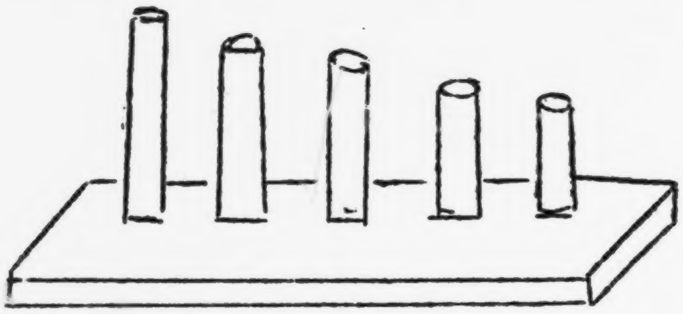
S. Yes.

T. If it did the same as for the sticks, the 3-stick rule would say $\bigcirc > \square$. But that is not true in the game, because $\square > \bigcirc$ as paper wraps stone. So we could not use our old 3-stick rule in finding out which of paper, rock or scissors wins the game.

The contrast was not clearly seen by all the students at this point. But they had become restless and a change in activity was called for. If sufficient time had been available, the topic should have been returned to once more. As mentioned in Section II, at a second round one would develop the contrast very systematically. Each of the scissors, rock and paper would be carefully identified with a specific colored stick, the results listed in pairs and the contradiction emphasized.

No assertion was made to, or asked for from the children as to whether the signs $>$ or $<$ should be reserved for cases where the transitive property holds. It did not seem relevant at this stage to tell them that mathematicians have reserved the symbol for such use, as they would not be using the symbols in an abstract context for some time.

The class then turned its attention to a wooden board, in which colored pencils of various lengths were standing in shallow holes.



The teacher asked about their arrangement. The students answered, "They are lined up in a special way." "They are lined up by color." "The yellow is smallest, the black is bigger than the yellow, the green is bigger than the black (etc)," and finally, "They are in a special way, the way they get bigger and bigger next to each other."

The teacher then told the class that he had a piece of licorice, which they would not all see. That they were to try to find out where the licorice belonged in the row, with as few questions as possible. That each question had to be of the type, "Is the licorice taller than the yellow pencil" or perhaps, "is it shorter than the black pencil?" That only one student would see the licorice, and in secret, compare it to the stick named and answer yes or no.

The answers would be listed on the board. The first person to know the answer would get the licorice.

The first time the questions of the class and the answers of the checker proceeded in this way:

Q. Is the licorice bigger than the yellow pencil?

A. The yellow pencil is smaller (marked $\overset{\text{yellow}}{O} < L$).

Q. Is the licorice bigger than the green?

A. The green is bigger ($L < \overset{\text{green}}{O}$).

T. We now know that it is bigger than the yellow, and smaller than the green. Do we know if it is bigger or smaller than the black?

S. No.

Q. Is the licorice bigger than the black?

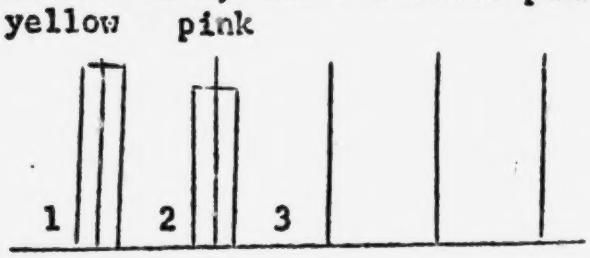
A. No. ($\overset{\text{black}}{O} < L$).

S. It goes right here (points between yellow and black).

In the next round the first question was, "Is it smaller than the blue?" In both rounds, the students started at an extreme. The game was left to be continued in the next session.

Session 9, July 23, 1964

Mimeographed sheets were passed out, with five vertical lines rising from a horizontal base, and the students were asked to paste a yellow strip on the first line (starting at the base) and a shorter pink strip on the second line (see figure).



To place the strips accurately enough a few children needed to repeat the procedure. The students were told that the teacher had a green stick that he wanted to put on the base so that the sizes would decrease in order, from left to right.

T. How many possible positions are there? If the green stick is in order in position 3, is it bigger or smaller than the pink? (Same question for position 2 and 1). Where would it go if it is bigger than pink and smaller than yellow? Now ask me whether it is bigger or smaller than one of these sticks.

S. Is it bigger than pink?

T. No, it is smaller. How many people know where it should go? (Writes $\overset{\text{green}}{\bullet} < \overset{\text{pink}}{\bullet}$).

S. (Several answer correctly).

T. I have a new green stick. Find out where it goes.

S. Is it bigger or smaller than yellow?

T. Smaller (writes $\overset{\text{green}}{\bullet} < \overset{\text{yellow}}{\bullet}$). How many of you know where it should go?

S. a) It can go in 2 (actually pointing at the position). b) It can go in 3. c) Is it bigger or smaller than both the pink and yellow? d) Is it bigger or smaller than the pink?

T. It is bigger than pink ($\overset{\text{green}}{\circ} > \overset{\text{pink}}{\circ}$) (Everyone knows where it belongs.) It was very easy with two sticks in place. You needed no more than two questions. Let us see what happens when we have 5 sticks (gold, white and blue sticks are added in descending order). Does anyone think they could find where my green stick belongs if they had only three questions? Let us count the number of places the green stick might go. (They point at the six different positions).

S. Is it bigger than blue?

T. Yes. Which places could it be in now? (They mark the five places left of blue).

S. Is it smaller than white?

T. No, bigger. What places can it be in now? (The students mark the three remaining places). Can it be between the blue and the white?

S. a) No, because there is no line there. b) No, because it is bigger than white. c) Is it smaller than gold?

T. Yes the green is smaller than gold. (The inequalities were noted on the board. The children said they now knew that the green stick should go between the white and the gold). How many questions did you use?

S. Three

T. Yes, but you were lucky the green belonged where it did. After the last question, there were still three possibilities. The way it was going, you might have needed five questions.

At this stage the children were well aware of the rules of the game and the implications of each answer. But they showed no awareness of a strategy in choosing questions.

Session 10, July 27, 1964.

To avoid confusion between yellow and gold, the tallest stick was replaced by one of green color, and the "unknown" stick was black.

T. What type of question can you ask in this game?

S. a) Is it bigger than pink? b) Is it bigger than yellow?

The class was divided into three teams, each team being challenged to get the result in no more than three questions. The first team proceeded as follows:

S. Is the black bigger than the green?

T. No, smaller.

S. Is the black bigger than the pink?

T. No, smaller.

S. Is the yellow bigger than the black?

T. No, smaller.

S. I know where it goes!

For a different black stick, the second team asked questions.

S. Is the black bigger than the white?

T. No, smaller.

S. Is it bigger than the blue?

T. Yes.

S. It is bigger than blue and smaller than white, so it goes between them.

Then it was the turn of the third team.

S. Is it smaller than green?

T. Yes.

S. Is the white stick smaller than the black stick?

T. Yes.

S. Is the black stick smaller than the pink?

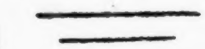
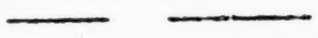
T. No, bigger.

S. Then it has to be between the green and the pink.

Only two or three questions, in this third session using this game, indicated an emergence of strategy.

Session 11, July 30, 1964.

Some students asked questions about gravity in outer space. A brief discussion followed, illustrating that children have learned much from television reporting of space events.

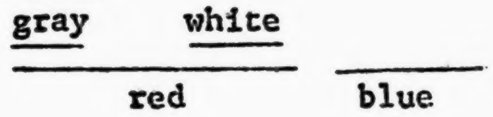
The teacher then turned the children's attention to the "addition" of sticks. What would they do to "add" the lengths of sticks? They were given the wooden dowels (tagged with colored paper). They were asked to lay the sticks out on the floor as they thought fit. Some children suggested  and others  . The teacher said that the class would see what rules they could find using the latter method.

After noting that red > gray and blue > white, they teacher asked which would be bigger , red and blue, or white and gray added together.

S. Red is bigger than all of them. (Others agree. The teacher had not noticed that red > gray + white,)

T. Who can actually add them, and then compare the added lengths?

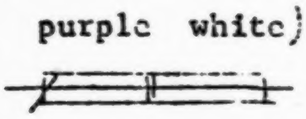
One child volunteered and succeeded, after some difficulty, in laying them out as follows:



Then pairs of children compared their sticks. The one with the larger stick would "add" to the larger stick of another pair; the smaller of two pairs likewise adding. Then they compared the added lengths as above. The obtained cases where the sum of the two smaller was larger than the largest single stick, and where one of the "smaller" was larger than one of the "larger", nevertheless the sum of the "smaller" was always smaller than the sum of the two "larger". This was repeated a few times. It was an awkward process requiring all four sticks to be held in position simultaneously.

T. It is difficult to arrange so many sticks at once. Let us try to find a way of remembering the length of each. Then we can put it away while adding or comparing the next. I am drawing a straight line on the board, and marking a place on it with an x (x————). Will someone mark off the length of their stick starting from the x? (A student lays a stick on the line and is asked to move one end to the "starting point." A chalk mark is made at the other end.) What can we write to remember that it is the blue stick that ends here? (Another student puts a blue patch above the segment). That is a very good way. Let us move over the patch to be over where the stick ended. Then the length of the blue stick won't get confused with shorter ones we may mark. (Other children were asked to check the end point for the blue stick). What can we call the line?

S. A measuring line. (A student added a white stick after a purple, on the line. It was done well, holding both sticks simultaneously on the line.



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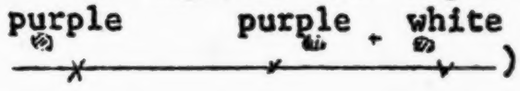
T. This time after the purple is marked off, I will take it away before giving the white. Who can add them together that way? (Two students failed, then a third succeeded



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After the difficulty the children had had in arranging four sticks on the floor, the utility of the "measuring line" as an accounting device seemed evident to them.

Session 12, August 3, 1964.

The children were given mimeographed "measuring lines" and a green and a red paper stick each. They were asked to mark off green from the starting point. Most marked off to the right of the starting point, but not all. The teacher asked them all to go to the right when adding a stick. They were then asked to add the red stick to the green. One student marked off the red from the starting point. The others added correctly, but some had forgotten the connection of these operations on the line to "putting two sticks together."

There was ambiguity in the marking of endings with color patches. To distinguish the end position of "red" from that of "green and red" they were asked to use both color patches with a plus sign between them, for the latter. They were next asked to start with the red and add green to it.

S. a) So it's backwards! b) It came to the same place!

T. Now add the green stick to itself. (Some students were at a loss. One student did it at the board and others followed).

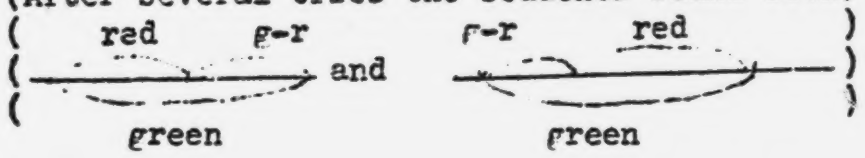
S. I need another green stick.

T. Why?

S. a) To add it to itself. b) He only needs one. (The first student then managed properly with one stick).

T. Find red + red + red. (With a little trouble a student accomplished that at the board).

T. Find out what is left over if the red length is removed from, or subtracted from, the green length. (After several tries the students found both)



Session 13, August 5, 1964.

During this session the children added and subtracted sticks in various ways, both at their desks and at the board. They performed the operations both by holding both sticks together and by marking one at a time on a "measuring line". They added sticks in both orders. For subtracting they first compared sticks. Finding out which was the smaller, they subtracted it from the larger. They discussed the process of subtraction in terms of chalk left over after using, or giving away, a certain amount. (It may well make more sense to deduce how much was used up by subtracting the left over piece from the standard piece).

The majority of the class attained proficiency in the addition and subtraction of sticks, and in the use of the"measuring line".

Session 14, August 7, 1964.

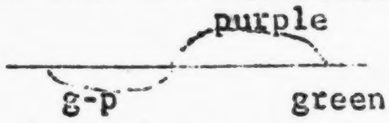
The children were asked to add and subtractvarious pairs of sticks at the board. They were asked to describe their procedure in detail; so that the operational approach would be fixed when they were given the problem of subtracting a larger stick from a smaller one.

T. What did you do differently in subtracting the red from the green, compared to when you added the red to the green? When you added the red, where did you start it? Which direction did you place it along the line from the starting point?

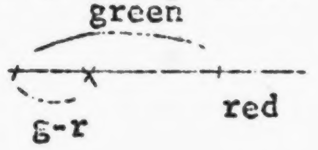
S. I started from the end of the green and pointed it to the window side.

T. When you subtracted it....?

S. I started it from the same place, but I pointed it to the door. I put it over differently.

T. (After several more students had subtracted smaller sticks from larger ones on a "measuring line" 

). Now subtract the green from the red. What is funny about what we did?

S. (Marking 

We end up on the other side of the starting point.

This was the last session, no further development of negative quantities was possible. However, the stage had been reached by many of the youngsters of using direction as well as magnitude to describe a distance. The operational methods of stick operations used had made this natural.

Comments on the Morse School results:

The above sessions indicated that most aspects, of the unit being developed here, were significantly more difficult for these pre-first grade children, than for the Estabrook children. The latter had half of their first grade training and maturing behind them, when starting to work on this material. The effect of their lesser experience showed up, for the Morse School children, as slower progress and more frequent loss of attention of some members of the class. However, when new teaching techniques were found overcoming specific deficiencies in preparedness, much of the performance difference disappeared. A simple but effective example of such a shift in technique was the use of colored patches in place of initials as stick length symbols.

Among the extensions of the unit initiated in these sessions, the use of deduction to derive n stick ordering from the transitive rule went very well. On the other hand, the application of ordering to the optimizing of information gained from a measurement (finding the order of a stick among several sticks of different length by comparing it to only a few of them) was only slightly developed by these children.

IV. Estabrook, 1964-5 school year.

In December 1964, weekly sessions were initiated with a superior class of second grade children. The same teacher who had instructed the first grade class in inequalities, in the spring of 1964, Miss Marie Mortimer, volunteered to instruct this older class. It was first hoped that a second grade class largely formed of students from the first grade class could be instructed. Various organizational changes made this impractical and only four of those students were in the new class of about thirty. Consequently, it was not possible to continue the unit sequentially, covering only new ground.

Several interesting aspects of investigation were presented by this new class. For the large amount of material that would only vary in detail from that of the previous year, Miss Mortimer would be relying for the most part on her own earlier experience and observations. The degree of her success in teaching these sessions would indicate the transferability of this material to a normal school context. Although she frequently had brief conversations with the author, there were no texts or teachers' manuals, nor even extensive notes on the previous year's material.

The greater age and experience of the students permitted evidence to be gathered with respect to: a) the relative rate at which theyexplored the same material that the younger children had explored; b) the change of interests and attention span with age; c) change in qualitative aspects of their learning capability, such as in relation to deduction and strategy planning. The question of applying mathematical results to planning strategy was of particular interest, in the light of the failure in this regard of the very young Morse School students. Such applications can be a powerful learning tool, supplying motivation, fun, deductive and inductive challenges well suited to the discovery approach. At what level of experience can the curriculum take advantage of those aspects?

In addition to improvements in presentation based on previous experience, this new class also allowed, d) the development of techniques based on knowledge these children had already gained in school, for example their knowledge of set concepts could be used. The students had been following the "Greater Cleveland" series.

Finally, e) the greater total time available and the presumed faster rate of learning would enable these children to go on to new material. The extension of their base to further inequality and arithmetic results could be developed and observed. Miss Kay Dillmore, who was responsible for the mathematics instruction in first and second grade, collaborated in the development and teaching of the new material, in the last part of the school year. The new work included the development of positive and negative fractions, and their addition and subtraction. Also there was two dimensional co-ordinatization and multiplication; multiplication of fractions and of several digit numbers (by grouping). The students were also introduced to the symmetry material of Dr. Marion Walter, during their last sessions.

We provide here only a slight elaboration of report on these sessions provided in the progress reports of the Estabrook project (see CCSM Report #33). The purpose in restating it here is to extract it from the reports on other Estabrook classes and put it into the context of the related experiments, commentaries on purpose and conclusions.

On December 18, 1964, I instructed the first session. We discussed the comparison of numbers, of elephants with giraffes, and so on. Many bases of comparison were introduced (heights, lengths, weights, etc.) and shown to lead to different results. An operational definition of shorter and longer sticks was requested ("How would you check to see?") and it was immediately proposed to "put one end of each stick together and see which stick has a piece left over at the other end." As they had had sets, they were asked to compare two sets and find out which one had more elements, without counting. After discussion and trials they came to the one-to-one check-off procedure. The set having elements left over, after the other was exhausted, was the larger. We several times, reversed the question from "which is bigger," to "which is smaller" to develop the $A > B \Rightarrow B < A$ rule.

In January, 1965, Miss Mortimer taught three of the sessions and I the fourth. The students proceeded at a rapid pace. The method of comparing sticks was likened to the comparison of sets by one-to-one correspondence. The "two stick rule" ($A > B \Rightarrow B < A$) was discussed in that context. Colors on the sticks were switched to show that the rule was independent of labelling.

The "three stick rule" was discovered through the following "warrior game". The Green and Red Knights were enemies who kept the size of their weapons (green and red sticks) secret from each other. However a neutral, the White Knight, was allowed by the others to compare his weapon with theirs. With the information of the White Knight the class had to decide if they knew which of the enemies had the bigger stick. They quickly realized that the neutral's stick had to be bigger than one of the red or green, and smaller than the other to be of help. This evolved to the requirement that when the inequalities were written so that the comparison stick was "sandwiched", the two inequalities had to have the same "order" to result in a "three stick rule": $A > B$, $B > C \Rightarrow A > C$ or "if (A,B) and(B, C) are in the same order, so is (A,C)". They filled out worksheets #4 and #5 to check their adeptness at recognizing and applying the two and three stick rules.

After making some comparisons among four sticks, they were asked to construct a four sticks rule. They set up the relations $A > B$, $B > C$, . $C > D \Rightarrow A > D$. They then found the two "three stick rules" within that statement; $A > B$, $B > C \Rightarrow A > C$ and $B > C$, $C > D \Rightarrow B > D$. From the first result, $A > C$, and

$C > D$ they were able to show (by again applying the "three stick rule") that $A > D$ followed. The generalization to "five (and more) stick rules" followed easily.

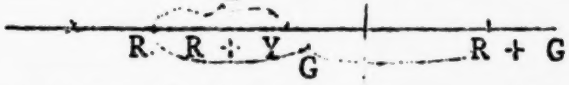
They enjoyed the example of the "paper, scissors, rock" game, as a case in which the order relations (the three stick rule) did not apply. They seemed to recognize the contrast clearly. They started to apply the order relations to the game of strategy, in which an unseen stick has to have its length located among five given ordered sticks, using the fewest possible $<$ and $>$ type questions.

In February, the children played a version of the strategy game which prevented the lucky location of the correct position in less than three questions. "Criminals" (of all six possible heights to fit among five sticks) were able to secretly pass a diamond between them. The detective could ask three $<$ or $>$ questions to locate the criminal with the diamond. Before answering the "mastermind" for the criminals could give the diamond to any of them, after which a truthful answer to the detective's question was required. (Searching for Uranium with a Geiger counter would eliminate the need for voluntary truth on the part of criminals!). After each question the eliminated set of criminals were separated, and could no longer receive the diamond. Before answering the next question, the "mastermind" could place the diamond with any one in his remaining band. Several students quickly found the secret of the three questions. "Masterminds" from the class were allowed to challenge class detectives until most members of the class had discovered the secret by themselves. Several were able to explain the strategy well.

The game was extended to seven and eight sticks. In the latter case, they quickly found that they needed one more question. Although the strategy of splitting in two was clear to this class, it was not clear to what extent the connection to the "three stick rule" was realized.

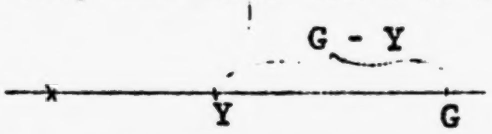
In the last session of the month, they developed methods of adding sticks and marking lengths on a line on paper, to facilitate comparison. They predicted and then checked that if $A < B$ and $B < C$, then $A + B < A + C$. During this month, Marie Mortimer instructed two of the three sessions, and I the other.

In March, the students compared two sticks and considered the addition of an unseen third stick to each. They showed a quick response and understanding of variations on $G > Y \Longrightarrow (R + G) > (R + Y)$, for different size R. While checking their predictions they gained facility with addition on a "stick line" and its notation. They used both the blackboard and mimeographed sheets (with paper strips for "sticks").



They also compared $G + R$ with $Y + R$ and $G + R$ with $R + G$ mentally, and then checked.

They were asked how they would find out how long a piece of stick was left over if they took away Y from G. They did the obvious with sticks only, putting two ends together and indicating the excess length of one at the other end. They were then asked to use the stick line. The response was both



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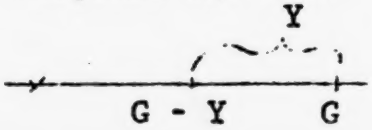
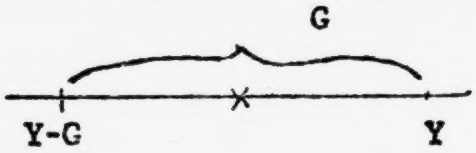


They observed that the second way was more in accord with their method of addition, i.e., the second stick started from the end of the first stick. After several such additions and subtractions, the class was asked to subtract a larger stick from a smaller. Many voices were raised to say "you can't do it", but the majority was not impeded by that consideration. They were only told to try proceeding as before. Then, from different parts of the room one heard, "I've got it". The whole class eventually discovered



T. What is different?

S. It falls on the other side of X.

T. Using your fingers compare the distance of X to (G-Y) with that of X to (Y-G).

S. They are the same length.

They went through other examples, including $[(G-R) + Y]$ and $[(R + Y) - G]$. They compared (Y-G) with opp (G-Y). By "opp" they meant to flip over about X. A reminder of the transitive pattern in $A > B,\ B > C \Rightarrow A > C$ was required, before they could make the transfer to the following use of the transitivity of equality: opp $(G-Y) = (-G+Y)$, $(-G+Y) = (Y-G) \Rightarrow$ opp $(G-Y) = (Y-G)$.

They then considered lengths of chalk.

T. We receive a certain length for the week and use varying amounts each day. How much do we have left over at the end of one week? How much do we have to borrow at the end of another week?

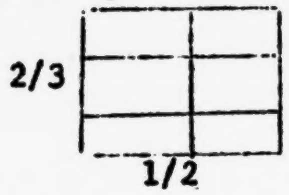
S. We borrow a piece big enough to bring us back to the starting point, from where we ended up on the negative side.

The class was given the length received and the length used each day. They were to report when they reached a day on which it was necessary to borrow chalk. About half the class was correct the first time around. Miss Mortimer taught three sessions. I taught one and observed another.

In April, Miss Mortimer continued the discussion of chalk lengths obtained, used and borrowed. This time the children started off the week in debt (on the negative side of the starting point). They added and subtracted the different lengths (G, Y, and R sticks of odd sizes) corresponding to chalk obtained from the office and chalk used. The children made it known when they were first "out of debt." The concept of using a fixed amount of chalk each day was introduced and the children asked to mark off the total they would have used by Monday, Tuesday, Wednesday, etc. When asked how they would mark these positions they responded with M, T, W, T, F; 1Y, 2Y, 3Y, 4Y, 5Y, (the yellow stick was used) and also 1, 2, 3, 4, 5. The starting point became 0. They marked off the various amounts they would have to borrow and suggested -1, -2, -3, -4, -5. They had thus developed a "number line" in part. They were asked to use this "number line" in solving such problems as, "Two pieces of chalk were delivered Monday. If one piece was used each day, how much had been borrowed by Friday?"

They were asked how many halves were in a whole. They were then asked how they would find half of a standard or unit piece of chalk. One child suggestd, "folding in half," but another said this would not be done with chalk and suggested breaking and finding, by trial and error, a piece which would add to itself to reach 1 from 0 on the number line. This was taken as the standard. The students cut strips of paper to fit this criterion. They were allowed to use symmetric folding to decide where to cut, if they wanted to. With this 1/2 length they found the positions of 1/2, 3/2, -1/2 etc., on their number line. They were familiar with the notation 1/2, but were still developing the nomenclature and symbolism for the others at the end of this month.

From the beginning of May the second grade class was instructed by Miss K. Dillmore two or three times a week using CCSM material, in addition to my weekly contact. She concentrated on multiplication by the use of arrays and of rectangular areas. Arrays of dots on blank paper were used first. As the numbers became larger the students were given 1/2 inch graph paper and were asked, for ease of counting, to mark off -1, 0, 1, 2, etc. up, and also across. This elicited from several students "This is just the same as using two number lines crossing each other." They marked off rectangles, according to the numbers in a product, and counted the enclosed unit squares quickly, by twos, fives, etc. They were easily able to do multiplications such as 12 x 17 in this way. They noted that altering the order of the numbers only rotated the rectangles, so that the number of squares had to be the same. They also looked for "missing factors" - how many rows of 3 squares were needed to make 12. Asked for the products of 10 x 2, 10 x 4, 10 x 10, etc., they realized that these represented a class of questions they could answer readily. When they were then asked to graph 23 x 12 marking off blocks of easily determined numbers of squares, many made maximal use of products of ten. They marked off the two 10 x 10 blocks, the 10 x 3, the 2 x 10 and the 2 x 3 blocks. They readily multiplied integers less than 100 in this way. They used the same rectangle construction for multiplying fractions, symmetrically subdividing the unit square into a sufficient number of rectangles to determine the fractional area. For instance, if multiplying 2/3 x 1/2 they would first mark off a rectangle "2/3 up and 1/2 across" in the lower left hand corner of the unit square. Then they would symmetrically divide the square into 6 rectangles of 1/3 by 1/2. Noting that the original rectangle was made up of two of these smaller rectangles, they concluded that 2/3 x 1/2 = 2/6 of the unit square. Furthermore, they could count three pairs of the smaller rectangles, so 2/6 = 1/3 was another result.



After the above exposure the students found the Greater Cleveland exercises on multiplication too easy. As multiplication had been operationally the same for fractions as for integers, they did not have the usual difficulty in comprehending the relationship of the two ("Why is 1/2 of 1/4 the same as 1/2 x 1/4?" is a typical question of students.)

The development of fractions was continued on the number line, during the session I taught. A variety of games of strategy between paired opponents was played, when the line was marked in half units. A typical game was: starting at -1, each player in turn could choose to add 1 or 3/2, or to subtract 1/2. The subtraction was not permitted twice in a row, determining a finite game.

The first player to reach +2 won. This type of game, which they enjoyed playing at length, quickly developed their facility for adding and subtracting fractional numbers. They defined 1/3 (three of them made up a unit interval), marked off the line in thirds, and played a game analogous to the one above. They easily extended to defining and working with 1/4, 1/5, etc. In a game permitting jumps of 2/3, 1 and -1/2, they found that they had to divide the unit into sixths to handle all results, and found relations such as 2/6 = 1/3. They answered many questions on inequalities, such as "Which is bigger 3/5 or 4/5, 1/10 or 1/11, 8/9 or 9/10?" Both rules of order and the definition of fractions were involved in their reasoning. Only a few students had trouble with, for instance, the relative size of 1/10 and 1/11. Other students explained that the pieces had to be smaller to get more into the unit interval.

For the last two weeks, new material was introduced. This material extended notions of congruency and discrete rotational symmetry into geometrical rather than arithmetical areas. Most of the class worked through Dr. Marion Walter's unit on the putting together of equilateral triangles, edge to edge. They observed the uniqueness, up to rotations, of shapes made in this way using two or three triangles. Many students were very systematic in exhausting arrangements. They discovered that they got different results if the laying of one triangle exactly on top of another was permitted, which maintained the edge-to-edge rule! Adding the rule that there could be no coverings, they found the three distinct shapes that could be made out of four triangles. They found they could fold these into tetrahedra or square pyramids. At the same time, with a small part of the class, Miss Dillmore guided the children through the sets of Dr. Walter's Mirror Cards. With the whole class, in the last session, I presented the class with Dr. Walter's unit of arranging up to five squares, edge-to-edge. They noted the symmetries and found new patterns rapidly. By the end of the session (and the school year), they had found about 8 distinct patterns of five squares.

Summary of the second grade experience:

These intelligent second grade students were clearly capable of handling all of the material presented to them. They were able to apply concepts to situations which displayed the power of those concepts at the same time as giving them practice with the manipulations. They were able to inductively apply the results to games involving non-trivial strategy. This, and their more explicit understanding of the key role of the transitivity axiom, differed qualitatively from the results with the younger classes. But their other accomplishments, e.g. deductive reasoning, group of concepts and definition (for inequality, addition and subtraction), and application to problems not involving strategy (thinking several steps ahead) - differed in speed of attainment rather than in final accomplishment.

The start they had made on the fundamentals of the number system seemed to offer a good base for further development in mathematics. This was demonstrated to a degree by the progress made, in the last few sessions, in their investigation of geometric concepts and arithmetical concepts and algorithms.

V. Conclusions

Before safe conclusions can be drawn concerning the proper place for this material in the curriculum, substantially more observation and improvement is needed on a small scale experimental level. For instance it is not known how much better the youngest children (Morse School age) may do with the better formulated versions of strategy games used with the second grade children. Again in retrospect it is clear that the two younger classes were not given sufficient time, or the best organization of material, to appreciate the contrast of the non-transitive "paper, scissors, rock" rules with the transitive length comparisons.

Thus the following inferences are certainly tentative. First, the total amount of time spent on this material in the first grade Estabrook sessions (Section II), or a little less, seems to be about right for that grade level. As concentration of material seemed to be an advantage at the Morse School, one may prefer to present the material over two different four or five week periods (two or three sessions a week), once in fall and once in spring. The first period should use some of the simplified presentation discussed in Section III, e.g. colored patches and cardboard $>$ signs. Because it is so essential to illustrating that the order structure is not trivial, the "paper, scissors, rock" game should be introduced in the first period, but should be given sufficient time and organization (including worksheets). Addition of sticks, subtraction, negative quantities and the beginning of fractions can be carried over between the two periods, with the more complicated addition and subtraction inequalities and addition of different fractions left for the second period. The second set of sessions in the first grade would seem to be the proper time to discuss the deduction of n - stick rules from transitivity. If in the course of first grade the children have learned about sets, they should be shown the analogy between their length comparisons and one-to-one correspondence methods. The games used in this grade, as in Section II, should not require extensive thinking ahead. The material of Section II on "signed sticks" (with arrows pointing from beginning to end or vice-versa) is perhaps better left for second grade, where it can be extended to two dimensions.

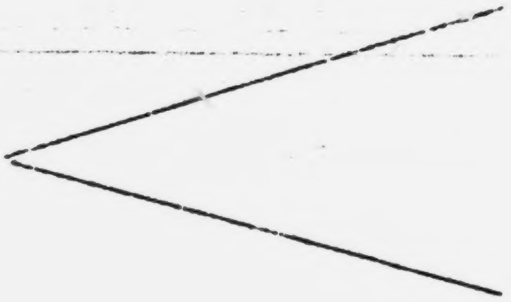
In the second grade the notion of signed lengths can be introduced together with the notion of "reversal" or "oppositing" (changing the sign). New rules of addition and subtraction then arise, and a discussion of associativity, commutativity and completeness can follow. For motivation, and to fashion facility in manipulation, several games of strategy can be introduced. The numbers that are explicitly named and calculable can be extended to all rationals. The co-ordinatization can be extended to two dimensions and multiplication of fractions and many digit numbers developed in that context. Division may be started in the same context. The first explicit discussion and use of algorithms of addition, and possibly of multiplication, should evolve within the second grade. Tentative discussion of irrationals should be possible also. On a different branch of mathematics, there can be a start on symmetry and vector concepts, anticipating or using material previously tried with third grade children (see CCSM Reports #33 and #34).

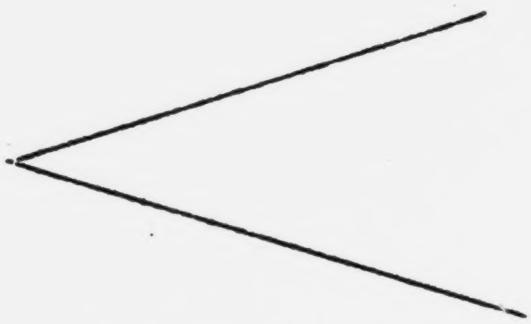
The above distribution of material between the first and second grade seems to the author to be consistent with the difficulties of the pre-first grade children and some topics, the rate of progress of the first grade children and with the ease with which the second graders handled most of the discussion. The combination of the above program with early geometry, intuitional symmetry, set concepts, and intensive development of the arithmetic of integers (probably including the use of different bases), would provide a base in first and second grade for the "goals for school mathematics" envisaged for later years. However, there are other mathematical topics, such as intuitional probability (see CCSM Report #37) which may require time in the same grades.

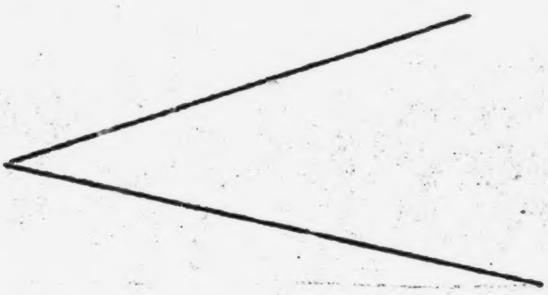
There seems to be little point in listing here those items of presentation which need further polishing, if not substantial alteration. Almost all the details need some degree of improvement. In particular prepared desk work (question sheets) should play a larger role ultimately than in this first experimental stage.

In my opinion the educational material of this report has proven sufficiently viable in the classroom, and of sufficient benefit to the students, to justify a new trial on a consecutive two year basis, keeping the same students throughout. There is a sufficient variety of presentation and material to justify teaching two or three groups of students in the same two year period. These groups could differ according to background or ability level. It is possible that the outcome of such a trial, by an appropriate team of teachers and mathematicians, would be material useful on a wide basis.

#3







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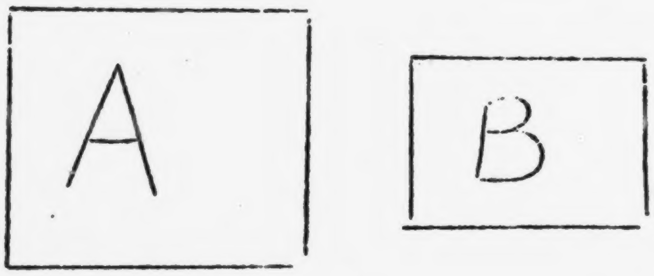

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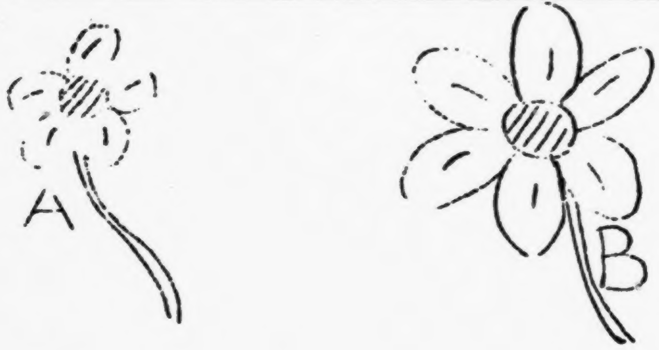

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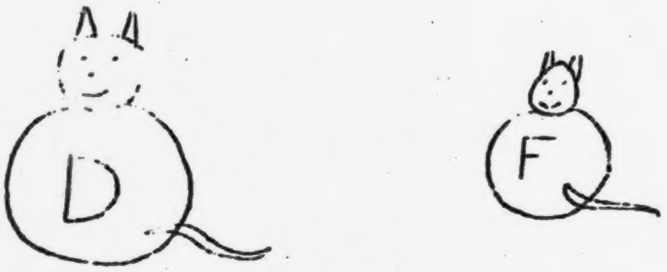

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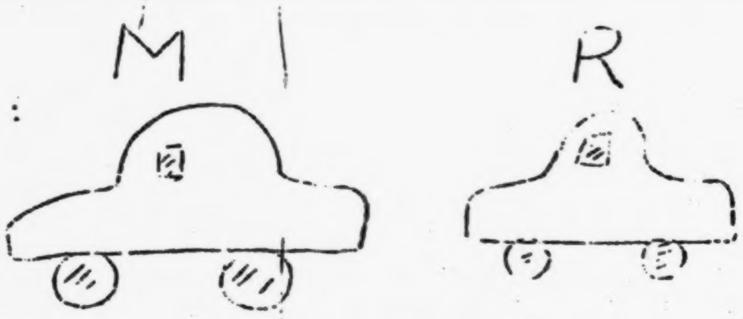

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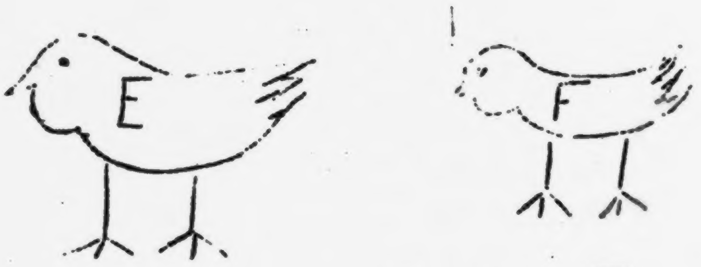

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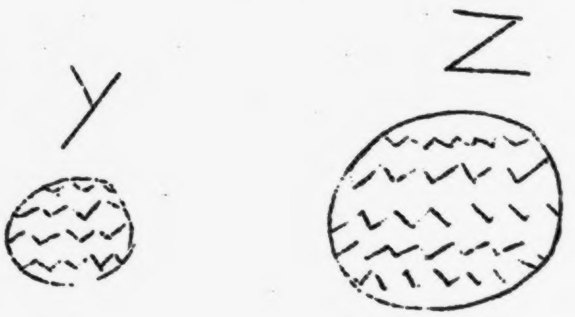

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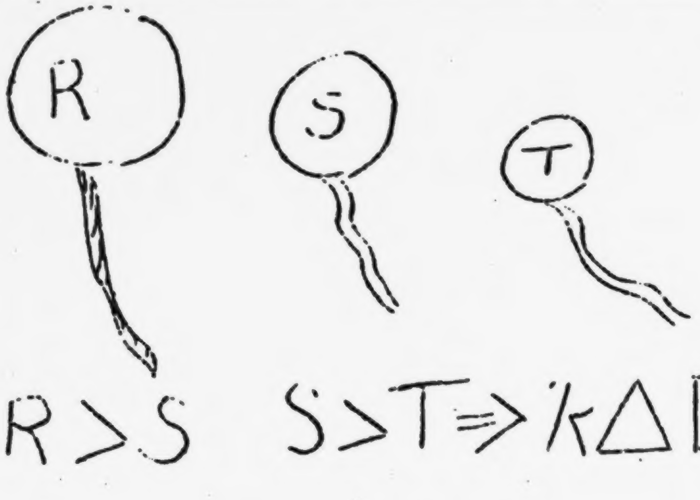

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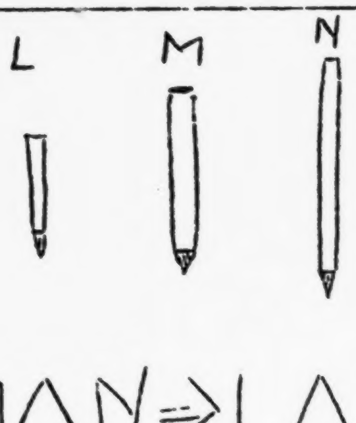

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